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## TECHNICAL PAPERS

AND

## DISCUSSIONS

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

# STATISTICAL ANALYSIS IN HYDROLOGY

By L. R. Beard,[1] Jun. Am. Soc. C. E.

Synopsis

The basic principles of the duration curve are reviewed in this paper in such a manner as to outline the various ways in which it is properly used in hydrologic analyses. Some of the details and differentiations discussed are of small significance compared to inaccuracies existing in the greater part of the hydrologic data available today. However, the proper use of the duration curve is essential to a fundamental understanding of statistical analysis and, since proper use is no more difficult than improper use, the differentiations are deemed relevant from an engineering standpoint.

## Conception of the Duration Curve

In its application to flood-control work, the duration curve may be looked upon as a refinement of the practice of basing the design on the largest flood of record. The difficulty of this old practice lies not only in the uncertainty that the maximum occurrence is representative for the period of record, but also in the absence of criteria for determining the variation of magnitude with period of occurrence. Both of these deficiencies are remedied by the proper use of the duration curve.

The theory of the duration curve is relatively simple. The occurrences of the future (those that are "in store" for the future) are considered to constitute a series, arranged in the order of descending magnitude. If there are an infinite number of occurrences in store, then this series may be represented by a continuous curve such as that shown in Fig. 1. Furthermore, if it is considered that there is no cycle or trend in magnitude with respect to time, it may be assumed that occurrences which happen within a given period have each been selected at random from the store of occurrences. Lastly, since the number of occurrences in store is considered to be infinite, the selection of a finite number of occurrences will have no effect on the distribution of magnitudes within the remaining series of occurrences.

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **February 1, 1943.**

[1] Asst. Engr., U. S. Engrs., War Dept., Los Angeles, Calif.

Since they have been chosen at random from the store of occurrences, the occurrences of record (those to be used in the analysis) may be used to estimate, by the laws of probability, the curve represented in Fig. 1. With this curve defined, forecasts concerning future occurrences may be made.

## Construction of the Duration Curve

The curve shown in Fig. 1 may represent all of the occurrences ever to happen or it may represent all of the yearly maximum occurrences ever to

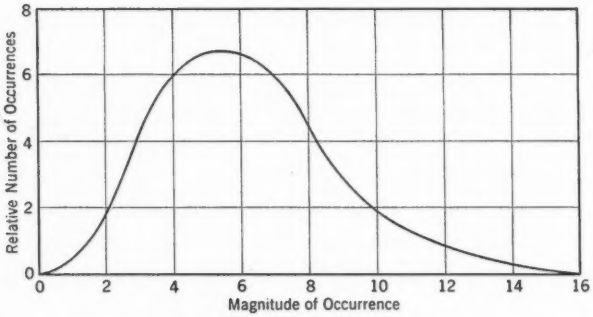



Fig. 1.—Distribution Curve

happen, or any one of many other series. This curve is called a probability or distribution curve and is similar in nature to the normal probability curve of Gauss and Laplace. Equal areas under the curve represent equal numbers of occurrences. The duration curve is the integral curve of the probability curve. Thus, if magnitude of occurrence is plotted against cumulative area under the probability curve, the duration curve is obtained. Fig. 2 shows the

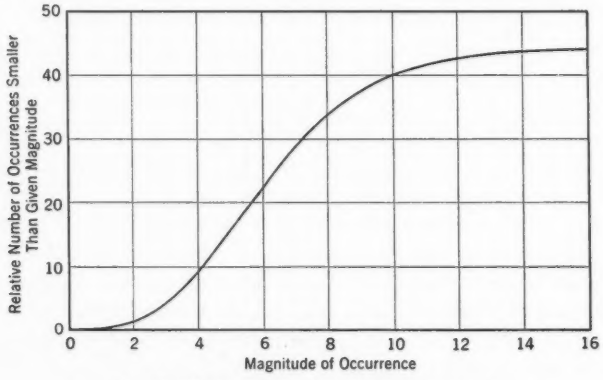



Fig. 2.—Duration Curve (Unconventional)

duration curve that corresponds to the probability curve shown in Fig. 1. The maximum value of the duration curve ordinate is equal to the total area under the probability curve. If this value is called 100%, then the ordinate

scale of Fig. 2 will range from zero to 100%, and this scale is called the percentage of occurrence scale. Convention interchanges the ordinate and abscissa scales and reverses the order of the percentage of occurrence scale such that the maximum ordinate corresponds to 0% of occurrence. The duration curve of Fig. 2 is shown conventionally in Fig. 3.

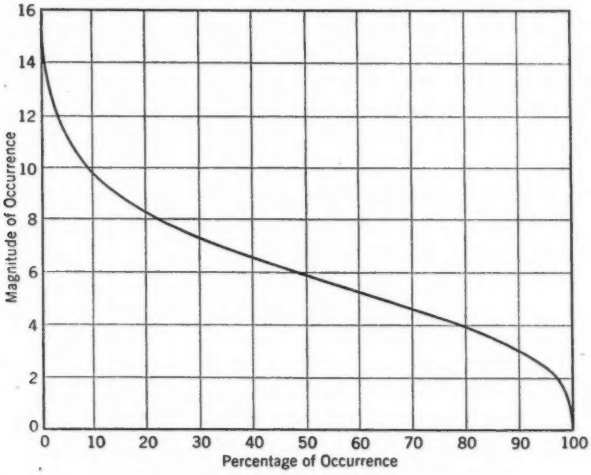



FIG. 3.—DURATION CURVE (CONVENTIONAL)

In constructing the duration curve from the occurrences of record, it is assumed that each occurrence is representative for the period—that is, the largest occurrence in a 10-yr period is considered to be the median of the largest occurrences of all of the 10-yr periods ever to happen. Likewise, the second largest occurrence of a 10-yr record is considered to be the median of the second largest occurrences from all of the 10-yr periods ever to happen, and so forth. Since the total number of occurrences greater than any median is equal to the total number smaller than the median, it is expected that half of the occurrences will exceed their medians and that half will be smaller than their medians. If each occurrence is plotted in the position of its median, a smooth curve drawn through the points will be the best available representation of the true duration curve.

In Fig. 3, all of the occurrences in excess of the 10% of occurrence value constitute 10% of the total number of occurrences. If one occurrence is chosen at random, the probability that it will exceed the 10% of occurrence value is 10%. Similarly, the probability that an occurrence chosen at random will exceed a given percentage of occurrence, $P$, is $P$ divided by 100, or $p$. If all occurrences are grouped at random in sets of ten, then the probability that the maximum occurrence of a set chosen at random will exceed the median of the maximum occurrences of all of the sets is 50% (since there are as many above a median as there are below). Therefore, it may be stated that the probability that none of the ten occurrences chosen at random will exceed the median of the maximum occurrences of all of the sets is 50%.

Let $P$ equal the per cent of occurrence of the median. Then $q$, or $(1 - p)$, will be the probability that an occurrence chosen at random will not exceed the median. The probability that ten consecutive occurrences chosen at random will be smaller than the median is therefore $q^{10}$ and is known to equal 50%, or 0.50. From this, the value of $q$ is 0.933, and $p$ equals 0.067. The percentage of occurrence corresponding to the median of the maximum occurrences from all of the sets is therefore 6.7%.

In a similar manner, it is possible to determine the percentage of occurrence that defines the median of all occurrences consisting of the second maximum occurrence from each set. In this case, the probability that at least nine of ten occurrences chosen at random will fall below a given value is 50%. The probability that none will exceed the value is $q^{10}$. The probability that exactly one will exceed the value is $10\,p\,q^9$. Then the probability that at least two of ten will exceed the value is one (certainty) minus the two foregoing probabilities and is known to equal 0.50. Thus,

$$1 - (q^{10} + 10\,p\,q^9) = 0.50 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

Since $p$ equals $1 - q$,

$$1 - (q^{10} + 10\,q^9 - 10\,q^{10}) = 0.50 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

and, therefore,

$$10\,q^9 - 9\,q^{10} = 0.50 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

A solution by successive approximations yields a value of 0.836 for $q$ and 0.164 for $p$. The percentage of occurrence corresponding to the median of the second largest occurrences from the sets is therefore 16.4. Values for the third largest, and others, may be computed similarly.

The "plotting position" of an occurrence is its estimated percentage of occurrence. Plotting positions for any set of ten arbitrarily chosen occurrences are given in Table 1.

TABLE 1.—PLOTTING POSITIONS FOR SERIES OF TEN OCCURRENCES

| No. of occurrence | Plotting position (%) |
|---|---|
| 1 | 6.7 |
| 2 | 16.4 |
| 3 | 25.8 |
| 4 | 35.5 |
| 5 | 45.2 |
| 6 | 54.8 |
| 7 | 64.5 |
| 8 | 74.2 |
| 9 | 83.6 |
| 10 | 93.3 |

## INTERPRETATION OF THE DURATION CURVE

In plotting the duration curve, the assumption was made that the recorded occurrences were representative of any set of events that would occur in a period equal to the period of the record considered. This is not true, however. The largest event may be twice as great as would normally occur in the given period, and the second largest may be smaller than the normal second largest occurrence. Throughout the entire series of occurrences, however, these discrepancies should counterbalance each other to a large extent. The longer the record, the more accurate is the balancing out. Since the series plotted is not accurately representative, it will not plot as a smooth curve. If the points that appear high were lowered somewhat and the points that appear low were raised in such a manner that they all lie on a smooth curve, the points then

would be a much better representation of a normal series of occurrences for a period equal to the period of record. In plotting the curve, the largest of ten occurrences was assumed to equal the median of the maximum occurrences of all ten sets. If the occurrence as plotted did not lie on the smooth curve, the assumption was in error, and the point on the smooth curve having the same percentage of occurrence was the new estimate of the median.

Suppose that the "store" of occurrences for the future is arbitrarily divided into sets of ten occurrences each. If one of these sets is chosen at random, there is a 50% chance that the maximum occurrence will exceed the median of the maximum occurrences of all sets. This is obvious since there are as many occurrences to choose from in excess of the median as there are smaller than the median. Thus, the median of the maximum occurrences of all of the sets is the smallest value that probably will not be exceeded in a single set chosen at random. This value for a ten-occurrence set is read on the smooth curve at 6.7%.

Since the smooth curve obtained is the best possible representation of the true duration curve, and since the curve is independent of the number of occurrences used in its construction, a normal series comprising any number of occurrences can be obtained from the curve. Thus, a normal series of one hundred occurrences can be obtained by merely picking off values at the respective points at which each of one hundred occurrences would be plotted (their plotting positions).

It often has been necessary to determine the frequency with which a given value is exceeded. If by the duration curve it is determined that the value in question has a 10% occurrence, then, on the average, 10% of the occurrences will exceed that value. Thus, if fifty occurrences happen in ten years, five can be expected to exceed the value.

## The Normal Duration Curve

The necessity of extrapolating duration curves of short-period records to make long-period determinations has arisen in almost every hydrologic investigation in which duration curves were used. Unless the curve can be described mathematically, the extrapolation is necessarily based upon a few occurrences at the end of the curve, and, therefore, is liable to large error. If the form of the curve can be determined beforehand, however, every point can play its part in determining the extrapolation.

Many attempts have been made to describe the duration curve mathematically. Of the many equations suggested, only one has a rigid mathematical derivation. This is the Gauss-Laplace normal distribution curve. The fundamental assumption upon which this law has been formulated is that the arithmetic mean of the variate is equal to the median of the variate. The justification of this assumption for many cases has been proved unquestionably by innumerable observed distribution curves. . However, the attempts that have been made to fit rainfall or runoff data to the curve have resulted in definite discrepancies. The result is that many individuals have taken it upon themselves to find a more suitable distribution curve to use in rainfall and runoff analysis. Unfortunately, the ordinary treatment of the problem has not been to attempt to discover the discrepancies in the Gaussian law and thereby cor-

rect the fundamental assumptions, but rather to adapt the law by imposing limits or using mathematical variations of the equation. Most of these do not adhere to the simple assumptions that form the basis of the law of Gauss. They do not have a rigid derivation, therefore, and will fit observed data only by accident.

Outstanding discussions of the problem here involved appeared in the *Transactions* of the Society in 1927[2] and 1936.[3] Particularly enlightening is the discussion by Arne Fisher.[4] Referring to Thiele, Mr. Fisher states:

> "He mentions that, if the distribution of the variate, $x$, as actually observed, is found to be skew, or to deviate from the normal probability curve, $x$ may be considered as a 'fictitious' observation of the 'pseudo-normal' variate, $t = h(x)$, which latter function is supposed to follow the Gaussian error law; that is, if $t = h(x)$ is distributed according to a known normal error law of exponential form, it is possible to find the frequency distribution of the inverse function, $x = g(t)$."

Again Mr. Fisher states:

> "Thiele's investigations led him to the obvious fact that if the geometric mean of the observations yields the most probable value, the arithmetic mean of the logarithms of the observations must yield the most probable value of the logarithms of the observations, so that if the logarithms of the observations are assumed to be normally distributed, a law of error can be found for the observations themselves by a logarithmic transformation of the variate. Such a law must necessarily lead to the geometric mean of the observations as the most probable value."

A logarithmic transformation of the variate often has been used with exceptional success. However, as is certain to happen when short-period records are analyzed, many variations from the ideal curve were observed. As it was not apparent that there was theoretical justification for using the logarithmic transformation, it was considered that other curves could be used which would have as good a basis. However, a substitute law seldom has been formulated that did not receive more adverse criticism than praise. Evidence supporting or refuting the applicability of the normal distribution curve to rainfall and runoff analysis is weak. To indicate the applicability or inapplicability of the normal distribution curve to rainfall and runoff occurrences would necessitate obtaining a good many 1,000-yr records.

When a series of rainfall occurrences consisting of every occurrence of record is plotted, the existence of the annual cycle may, and probably does, impart a degree of "skewness" to the series, since it cannot be stated that each occurrence is independent of every other. If the series selected consists of the maximum occurrence of each year, however, the series is not affected by the annual cycle. Such a series, having the advantage of a normal distribution, often can be used instead of the cumbersome series consisting of every occurrence recorded. If it is necessary to construct a duration curve of all occurrences, the duration curve of yearly maximum occurrences may be used to

[2] "Straight Line Plotting of Skew Frequency Data," by R. D. Goodrich, *Transactions*, Am. Soc. C. E., Vol. 91 (1927), p. 1.

[3] "An Asymmetric Probability Function," by J. J. Slade, Jr., *ibid.*, Vol. 101 (1936), p. 35.

[4] *Ibid.*, p. 72.

extrapolate the curve if extrapolation appears difficult, since one curve is a function of the other.

The writer has constructed duration curves for six rainfall records from southern California stations varying in length from forty-nine to sixty-two years. Four duration curves of 24-hr rainfall for these stations are shown in Fig. 4. The data were plotted on logarithmic-probability paper (on which a duration curve will be a straight line if the logarithm of the variate follows the normal probability curve). Some good results have been obtained, and some sizable deviations exist. This, however, constitutes neither a proof of the applicability of the law nor a condemnation of the law. The records are far too short to serve as a basis for testing the law itself. They do, however, illustrate the typical duration curve and show what type of deviations from the normal may be expected.

## Accuracy of the Duration Curve

The writer has attempted an analysis of the accuracy obtained in using a duration curve. After choosing six long-period rainfall records of stations in the vicinity of Los Angeles, Calif., duration curves of yearly maximum, daily rainfall totals were prepared for the entire period of each station. In addition to these, the periods were broken down into 10-yr, 15-yr, 20-yr, 25-yr, 30-yr, and 40-yr periods, and duration curves for each period for each station were constructed. The average period of record for the stations was about sixty years. Each station, therefore, had approximately six 10-yr records, four 15-yr records, three 20-yr records, two 30-yr records, and one 40-yr record. Duration curves were drawn as straight lines on logarithmic-probability paper.

The first step in the analysis was to determine for a 10-yr period whether the 10-yr normal maximum occurrence computed by the duration curves had greater correlation than did the actually recorded 10-yr maximum occurrences. The values obtained from the duration curves for a given station were averaged, and the percentage deviation of each value from the mean was computed. The thirty-three deviations thus obtained from the six stations were used to compute the probable error of a single observation. By a similar procedure, the probable error of a single recorded 10-yr maximum occurrence was computed. The probable errors thus obtained were 12.6% for a duration-curve value and 17.5% for a recorded value.

The probable errors for values of the longer periods could not be computed satisfactorily in the same manner since there were not enough occurrences for each station to establish a good mean value. However, probable errors for these periods were computed by assuming that the correct value for a given station is that indicated by the duration curve for the entire period of record for that station. Deviations were computed as percentages of this long-term value, and probable errors were then determined. These are shown in Table 2(*a*). It will be noted that both the probable error and the largest deviation experienced in the computations are consistently higher for recorded values than for computed values. Furthermore, as the period of record lengthens, the probable error of a computed value is greatly reduced, whereas that of a recorded value is reduced by only a small amount. It is apparent, then, that

one of the advantages of design by the duration curve rather than by the maximum occurrence of record is that greater accuracy is obtained.

To provide a better indication of the variation of accuracy with the period of record, the probable errors of a single 100-yr value, of a single 50-yr value,

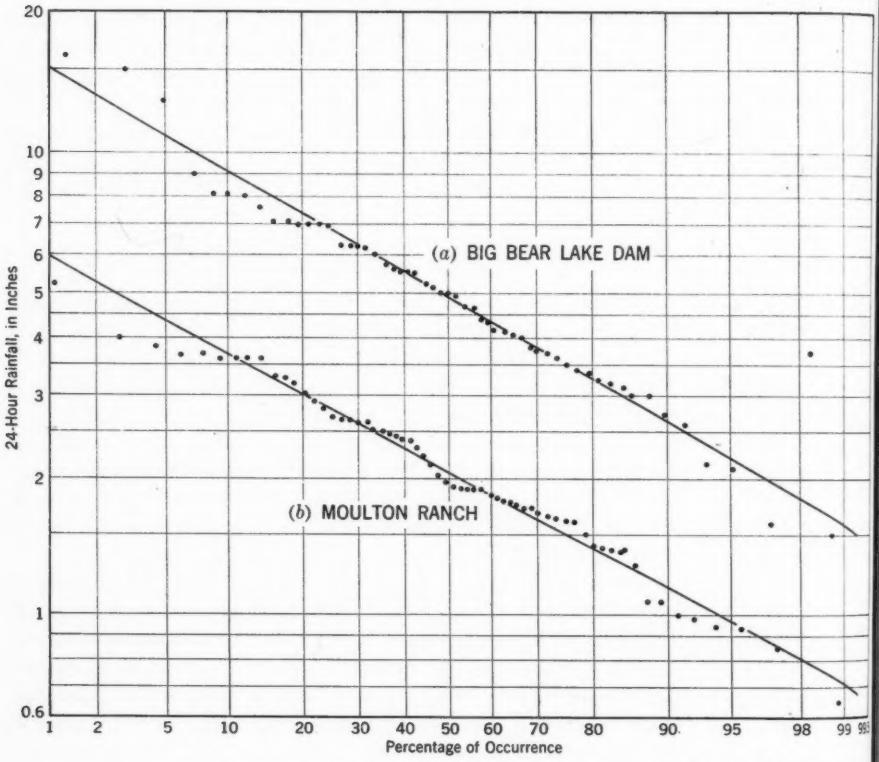



FIG. 4.—DURATION CURVES OF 24-HR RAINFALL

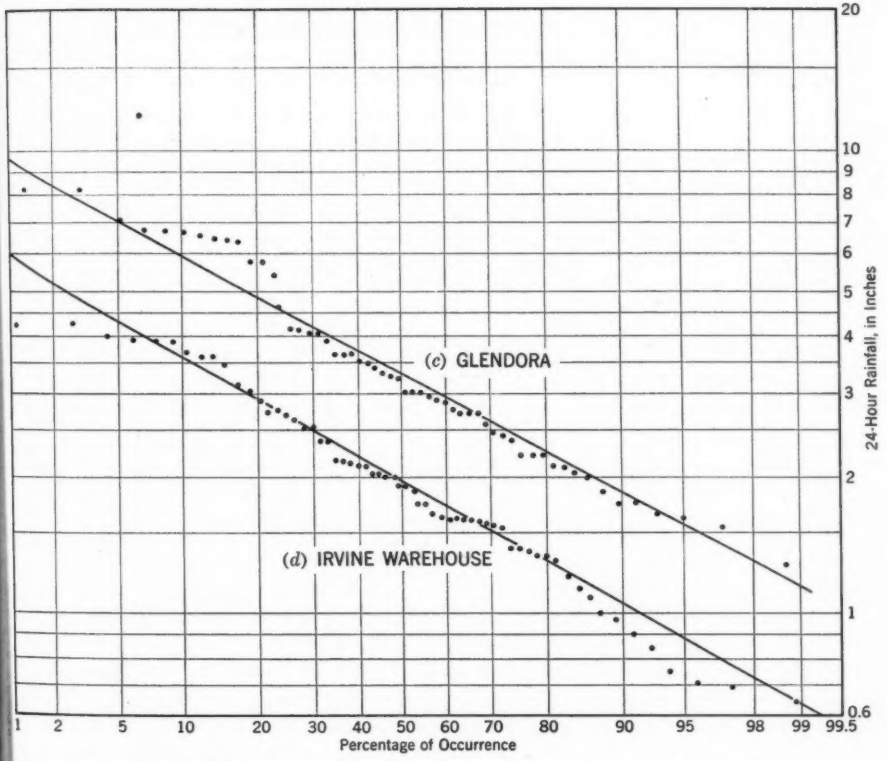
and of a single 1-yr value were computed for the same periods of record as in Table 2(*a*) and in the same manner. The results of these computations are shown in Table 2(*b*).

Table 2 applies only to curves having approximately the same slope as the curves of Fig. 4. If the slope of the curve is greater, the probable errors will be greater. A good index of the slope of the curve is the ratio of the 1% value to the 99% value on the curve. This ratio may be used in an analysis to relate probable error to the slope (degree of dispersion) of the duration curve.

## MAGNITUDES FOR DESIGN

In the case of an ordinary flood-protection structure, the period for which the structure is to afford protection is chosen by an economic analysis. If this is the case, the structure should be designed to control the normal sequence of

floods to be expected within the given period. If the economic side is of secondary importance, it may be found advisable to design the structure for an abnormally severe sequence of floods that have a certain degree of probability of occurring during the life of the structure. The extreme case is that




Vicinity of Los Angeles, Calif.

in which the structure is designed for the maximum possible flood which can take place within the lifetime of the structure.

A loosely used term in this regard is "maximum probable" occurrence. This term is not definable unless one states the degree of probability intended. "Probable," in the ordinary sense of the word, means greater than a 50–50 chance of happening or greater than 50% probable. However, some engineers think of "maximum probable" as much greater than this. One attempts to conceive of something possible, but having no probability of happening. This is impossible to do since zero probability is synonymous with impossibility. It would seem advisable in view of this misunderstanding to eliminate the phrase, "maximum probable," from the hydrology vocabulary.

In deriving the values shown in Table 1, the writer assumed that the value sought was that which has a 50% chance of being exceeded within a given

period. However, if the value sought, for instance, is that which has a 25% chance of being exceeded in one hundred years, then, instead of using possibly a value having a 50% chance of being exceeded in, say, two hundred years (as is often erroneously done), a value can be derived easily that will determine a flood having a 25% chance of being exceeded in one hundred years. This is: $P = 100(1 - 0.75^{0.01}) = 0.287$. Thus, it is not only inadvisable, but also unnecessary, in dealing with a structure to be designed for 100-yr protection, to consider any frequency other than the 100-yr frequency.

TABLE 2.—COMPUTED AND OBSERVED NORMAL MAXIMUM PROBABLE OCCURRENCES

| Years of record | (a) Probable Error and Maximum Deviation; Computed and Recorded | | | | | | (b) Probable Error of a Single Computation (%) | | |
|---|---|---|---|---|---|---|---|---|---|
| | Normal occurrence (years) | No. of curves | Probable Error (%) | | Largest Deviation (%) | | 100-yr occurrence | 50-yr occurrence | 1-yr occurrence |
| | | | Duration curve | Recorded | Duration curve | Recorded | | | |
| 10 | 10 | 33 | 13.0 | 17.9 | 33 | 67 | 19.1 | 17.1 | 12.5 |
| 15 | 15 | 21 | 13.0 | 17.7 | 35 | 51 | 17.5 | 15.9 | 7.8 |
| 20 | 20 | 15 | 11.2 | 15.6 | 37 | 42 | 14.4 | 13.0 | 4.5 |
| 25 | 25 | 12 | 5.7 | 14.0 | 16 | 33 | 7.0 | 6.2 | 3.2 |
| 30 | 30 | 11 | 6.3 | 13.0 | 18 | 32 | 7.4 | 7.2 | 3.4 |
| 40 | 40 | 6 | 4.2 | 14.3 | 10 | 30 | 5.0 | 4.3 | 2.6 |

To protect a dam against possible failure, it would be ideal, if feasible, to design a spillway for the maximum possible flood flows. However, it is found that this quantity, if estimated conservatively, is so huge as to prohibit design. In his paper entitled "Maximum Probable Floods on Pennsylvania Streams," C. F. Ruff,[5] M. Am. Soc. C. E., states: "The term 'maximum probable flood' is used to describe not the largest flood possible, but a flood so large that the chance of its being exceeded is no greater than the hazards normal to all of man's activities." This definition has been criticized as being too vague. However, the basic idea seems to be a good one and, if the probability so defined can be put down as a percentage probability such as 10% (one chance in ten), then the flood under consideration can be determined mathematically, although a large probable error may exist at such a remote point on the duration curve. This large probable error may cause some to hesitate to use the quantity, but there is no type of analysis available by which the error can be reduced. Therefore, it may be stated that a so-called "maximum probable" flood can be determined by the use of the duration curve if the degree of probability of occurrence within a given period of time is stipulated.

## DISCUSSION

In his paper entitled "Duration Curves," H. Alden Foster,[6] M. Am. Soc. C. E., states: "Having only a limited number of items from which to plot a con-

[5] *Transactions*, Am. Soc. C. E., Vol. 106 (1941), p. 1453.
[6] *Ibid.*, Vol. 99 (1934), p. 1220.

tinuous curve, the assumption is made that this continuous curve would pass through the center of each block of the actual record." It is apparent that this assumption is slightly in error, since the values thus obtained for Table 1 would be 5, 15, 25, etc., instead of 6.7, 16.4, 25.8, etc. As the number of items increases, however, the error in this assumption rapidly becomes negligible, and curves plotted by the approximate method will be very close to those plotted in the correct plotting positions. Since computation of plotting positions for various numbers of occurrences requires considerable time, it has been impossible to include a table of plotting positions. For the same reason, it may be found to be impracticable for an individual to compute plotting positions for certain studies. In this case, a good approximation may be made by computing the plotting position of the largest occurrence and estimating the plotting positions of the remainder of the occurrences, using for a guide the approximate plotting positions referred to by Mr. Foster and expressed by the equation:

$$P = \frac{2m - 1}{2n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

It will be noted that the true plotting positions are always closer to 50% than are the approximate ones, and that the difference between the true and approximate values becomes smaller as 50% is approached.

In using the duration curve for flood-control design, the design flood is usually defined as that which will be exceeded once (on the average) within the lifetime of the structure. Since common sense dictates that a flood which probably will be exceeded is ordinarily a poor design flood, it appears advisable to adopt the design flood definition as the minimum flood which will probably not be exceeded within the life of the structure. Table 3 gives percentage of occurrence values (one of which was determined in the foregoing analysis) of design floods for various periods for which structures may be designed.

TABLE 3.—PERCENTAGE OF OCCURRENCE VALUES

| Life of structure (years) | Percentage of occurrence (design flood) |
|---|---|
| 1,000 | 0.07 |
| 500 | 0.14 |
| 300 | 0.23 |
| 200 | 0.35 |
| 100 | 0.69 |
| 50 | 1.38 |
| 25 | 2.73 |
| 10 | 6.70 |
| 1 | 50.00 |

It has been common practice, in order to facilitate an analysis, to plot only the upper portion of a duration curve for which a great number of occurrences are available. From an engineering viewpoint, this procedure is generally justified because it is approximately correct and eliminates a great deal of work that in most cases is unnecessary. However, the construction of a curve from a partial series has two disadvantages—plotting positions are approximate, and a partial series cannot have a normal distribution. These disadvantages may in some cases (in which more accurate work is justified by design considerations and available data) outweigh the advantage of expediency.

Probably one of the most important weaknesses of the duration curve results from its use as a mechanical process, the accuracy of which is often overestimated and sometimes underestimated. One therefore hesitates to use the

curve for a record of short duration. However, in addition to the recorded occurrences, it is possible to use all types of rational comparisons and analogies to assist in constructing the curve itself. If this can be done properly, a very satisfactory design can be obtained. Any method used in hydrologic design must be flexible and adaptable to varying quality of data. Hydrology has not advanced far enough but that judgment must still play a major part in any analysis.

## Conclusions

Conclusions drawn from the preceding analyses and discussions are as follows:

(*a*) In order to represent best a distribution curve using a finite number of occurrences, the occurrences should not be plotted with equal spacing;

(*b*) Following from conclusion (*a*), the plotting of a series of occurrences above a base value as a complete series is not mathematically rigorous;

(*c*) The use of the Gaussian normal distribution curve has in no way been proved to be inapplicable to a properly chosen rainfall or runoff series;

(*d*) The duration curve can be used to determine the occurrence that probably will not be exceeded within a given period as well as that occurrence which will be exceeded (on the average) once within a given period; and

(*e*) The "maximum probable" flood, adequately defined, can be determined by use of the duration curve.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

# PERMEABILITY OF MUD MOUNTAIN DAM CORE MATERIAL

BY ALLEN S. CARY,[1] ASSOC. M. AM. SOC. C. E., AND BOYD H. WALTER,[2] AND HOWARD T. HARSTAD,[3] JUNIORS, AM. SOC. C. E.

SYNOPSIS

During the early investigation of the material contemplated for use in the impervious core of the Mud Mountain Dam (in a northwestern state), it was proved that the moisture content of the material when compacted had a much greater influence on the permeability than did the density of the material, within the range of densities at which it would be compacted during construction. Since the voids in the soil act as conduits for the water, it is postulated that the water content of the soil at the time of compaction governed the arrangement of the voids. At a high water content, the voids were apparently all of uniformly small size and the permeability was low. At a low water content, but at the same numerical void ratio or density, the voids were apparently of a wide range of sizes and the permeability was high. Thus, a wide or narrow range in void sizes may be likened to a wide or narrow range in particle sizes.

That the wide range or narrow range of void sizes may be imparted to the soil at will, with absolutely no variation in density being necessary, is proved by the extreme range in permeability among groups of samples compacted at very nearly identical densities but at different moisture contents.

This paper reports the results of an investigation inaugurated to determine the permeability-voids ratio relationships of the proposed borrow-pit material as a basis for the distribution of the material in the various zones in the Mud Mountain Dam. So pronounced were the effects of slight moisture differences on the permeability of the material that moisture control during construction

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **February 1, 1943.**

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became of paramount importance, not so much for the usual density control but primarily for control of the permeability of the core material.

Throughout the paper reference is made to "optimum water content" and "void ratio." In order to accommodate those who may not be conversant with this terminology, the following definitions are given:

> Optimum water content—"The water content at which the maximum density is produced in a soil by a specific amount of compaction."[4]
>
> Void ratio—"The ratio of the volume of intergranular space to the volume of solid particles in a given soil mass without regard to the proportions of liquid, air, or gas which may occupy the space."[4]

This paper will discuss in turn: (1) The material tested, (2) method and equipment, (3) test results, (4) interpretations, and (5) further investigation.

## Materials Tested

Test results in this paper apply only to the materials used in the impervious core of the Mud Mountain Dam. The soil is a mixture of stream gravel and glacial till. The average mechanical analysis of the material after removing all boulders larger than 6 in. is shown in Fig. 1. Curves 1 and 2 represent

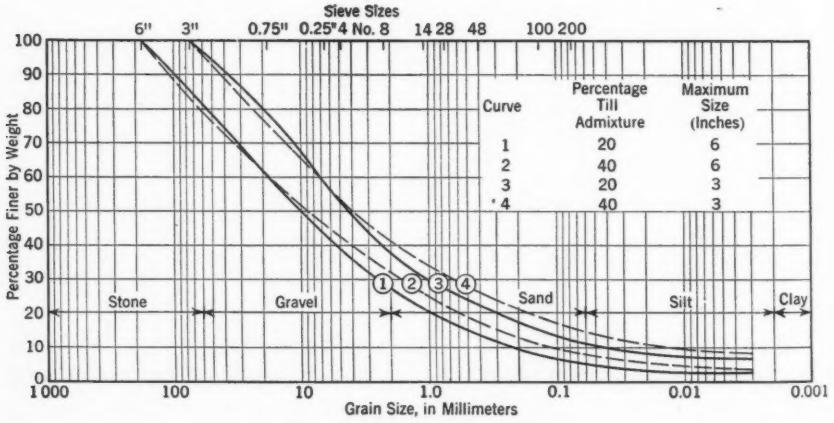


Fig. 1.—Mechanical Analysis

20% and 40% till admixture, respectively. The curves reveal an extreme range of particle sizes from colloids to 6 in.

The colloids are found primarily in the stream gravel and occur naturally as a coating around the particles in the deposit. They have been identified by the National Bureau of Standards by X ray and petrographic analysis as montmorillonite, an alteration product of volcanic glass and ash. Because of its extreme flatness, montmorillonite has been described as a "one-dimensional colloid." The extreme sensitivity of the mineral to excess moisture results

[4] "Soil Mechanics Nomenclature," *Manual of Engineering Practice No. 22*, Am. Soc. C. E., 1941.

in a marked loss of stability in the gravel when it is slightly above the optimum water content.

For all permeability tests, samples containing only those particles passing a 3-in. screen were used. Mechanical analyses of this material are shown in Fig. 1, curves 3 and 4. The range of void ratios at which the material was tested was limited to those expected during ordinary construction on the fill.

## Equipment and Methods

*Apparatus.*—Cylinders 13.54 in. in diameter and 12 in. high (1 cu ft) were used throughout the testing program for compaction and permeability. The permeameters were arranged with screens top and bottom to permit the free passage of water to and from the sample, and they were operated either as variable-head or constant-head upflow permeameters, as shown diagrammatically in Fig. 2. Except for very pervious materials, all tests were run with the variable-head hookup.

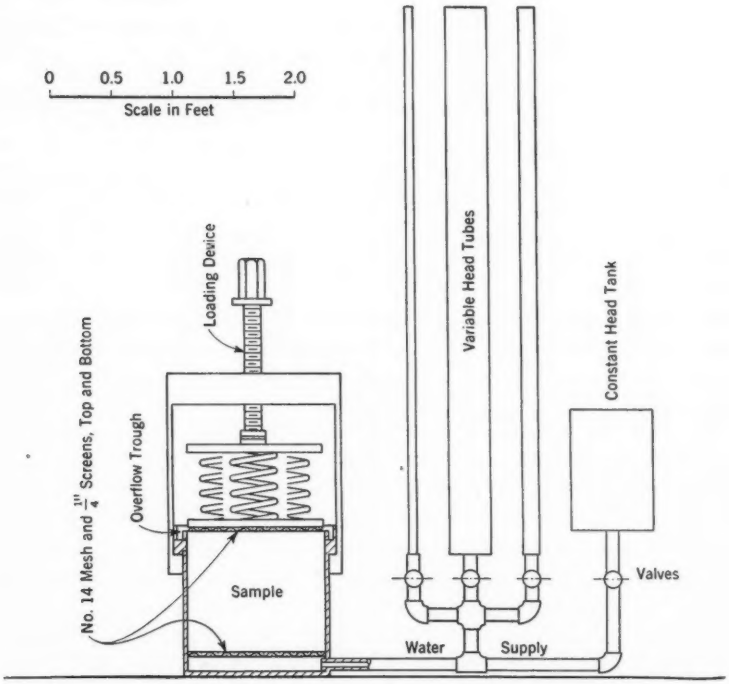


Fig. 2.—Part Section of Permeameters

*Compaction.*—The material was compacted in the compaction cylinder or permeameter in three layers with seventy-five 12-in. free-fall blows per layer of a 25-lb hammer with a 3-in. diameter face. This gave compaction which was equivalent to that obtained when the material was compacted on the embankment. After each layer was compacted, the surface was completely

scarified to break up any surface skin that might tend to reduce the permeability.

*Water.*—The water was distilled and preheated to above room temperature so that there was a falling temperature gradient through the sample during the entire run. This assisted in the removal of trapped air in the sample and hence in the prevention of air-locking.

An apparatus for de-airing water was assembled similar to that described elsewhere by G. E. Bertram.[5] It was impossible to obtain enough water to run all of the tests and, since preheating the water seemed as effective in this material, the de-airing was discontinued entirely.

*Test Procedure.*—A sample was compacted in the cylinder at the desired moisture content and loaded with 1 ton per sq ft. Water was started flowing up through the sample with a constant-head hookup as shown in Fig. 2. When the sample had become saturated, or rather when water appeared at the upper surface of the sample, the valves were turned for variable head, and duplicate permeability tests were run. For the long-time tests, the valves were turned back for constant head, and each 24 hr thereafter, for a period of from one to eight weeks, permeability determinations were made.

## Tests and Test Results

The specified moisture limits on the embankment were within 1% above and 2% below the optimum moisture. To allow for factors of uncertainty in field control and to determine any dangerous or difficult conditions that might be approached at the upper and lower moisture limits, laboratory tests were run on the material ranging from 4% below the optimum water content to 4% above.

Tests were run on the materials in four groups, as follows: Gravel with (1) 20% till admixture, (2) 25% till admixture, (3) approximately 40% till admixture from a test fill, and (4) core material with approximately 35% till admixture, sampled during construction of the dam.

*Group 1, 20% Till Admixture.*—A cubic yard of the core material containing 20% glacial till and 80% gravel was thoroughly mixed, and by successive quartering was divided into eighteen "identical" samples of approximately 160 lb each. Three of the samples were used to determine the optimum water content of the material. On the basis of this determination, three of the samples were placed in the permeameter at 4% below the optimum, three at 2% below, three at the optimum, three at 2% above, and three at 4% above the optimum. Fig. 3 shows the results of compaction tests plus the unit weights obtained when compacting the material in the permeameters at the various water contents.

As each sample was placed in the permeameter, it was saturated and the permeability test run. All samples were placed as nearly as possible in the same manner except that the water content was varied. There was, of course, some unavoidable variation in the void ratio.

[5] "An Experimental Investigation of Protective Filters," by G. E. Bertram, publications from the Graduate School of Engineering, Harvard Univ., Cambridge, Mass., Soil Mechanics Series No. 7, January, 1940.

When permeability is plotted against void ratio, on semilogarithmic paper, a straight-line relationship is usually apparent for most soils. In group 1, however, there is none of the usual straight-line correlation between the permeability and the void ratio of the entire series, as shown in Fig. 4(*a*). However, there is a good correlation between each set of three points obtained from those samples placed at the same water content. Fig. 5(*a*) shows a definite correlation between the permeability and the water content at which the material was placed.

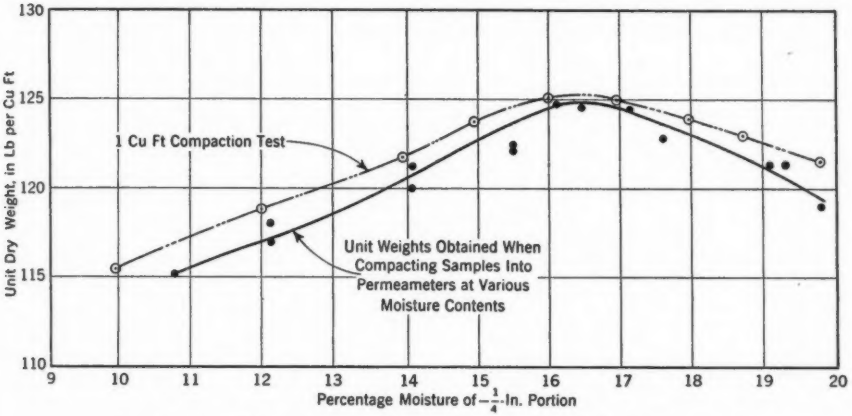


FIG. 3.—UNIT WEIGHT VERSUS MOISTURE CONTENT; 20% TILL ADMIXTURE

*Group 2, 25% Till Admixture.*—A similar series of tests was run for a 25% till admixture with the same permeability-moisture correlation as obtained in group 1. These tests are plotted, permeability versus water content, in Fig. 5(*b*). Permeability-void ratios are plotted in Fig. 4(*b*) and show none of the usual straight-line relationships.

*Group 3, 40% Till Admixture.*—A third series of tests was run using samples selected at random from a test fill constructed of core material with approximately 40% till admixture. The permeability-moisture correlation in Fig. 5(*c*) is better than might be expected when it is considered that this material varied considerably in till content and in mechanical analysis. Permeability-void ratios are plotted in Fig. 4(*c*) with the same complete absence of correlation.

*Group 4, Embankment Material.*—Fig. 6 shows the results of observations on 256 of the samples taken for permeability control of the core of the dam during the early part of construction, and directly from the borrow pit during excavation operations. The till admixture varied from 25% to 40%. All tests were run at the water content at which the samples were received in the laboratory. Here again, the permeabillty-moisture correlation was surprisingly favorable when it is considered that the mechanical analyses of the samples varied considerably.

It will be noted that, in both groups 3 and 4, there is a limited range of water contents. Had the range been wider, there is little doubt that the curves could be drawn as shown by the broken-line extension of the curve in Fig. 6.

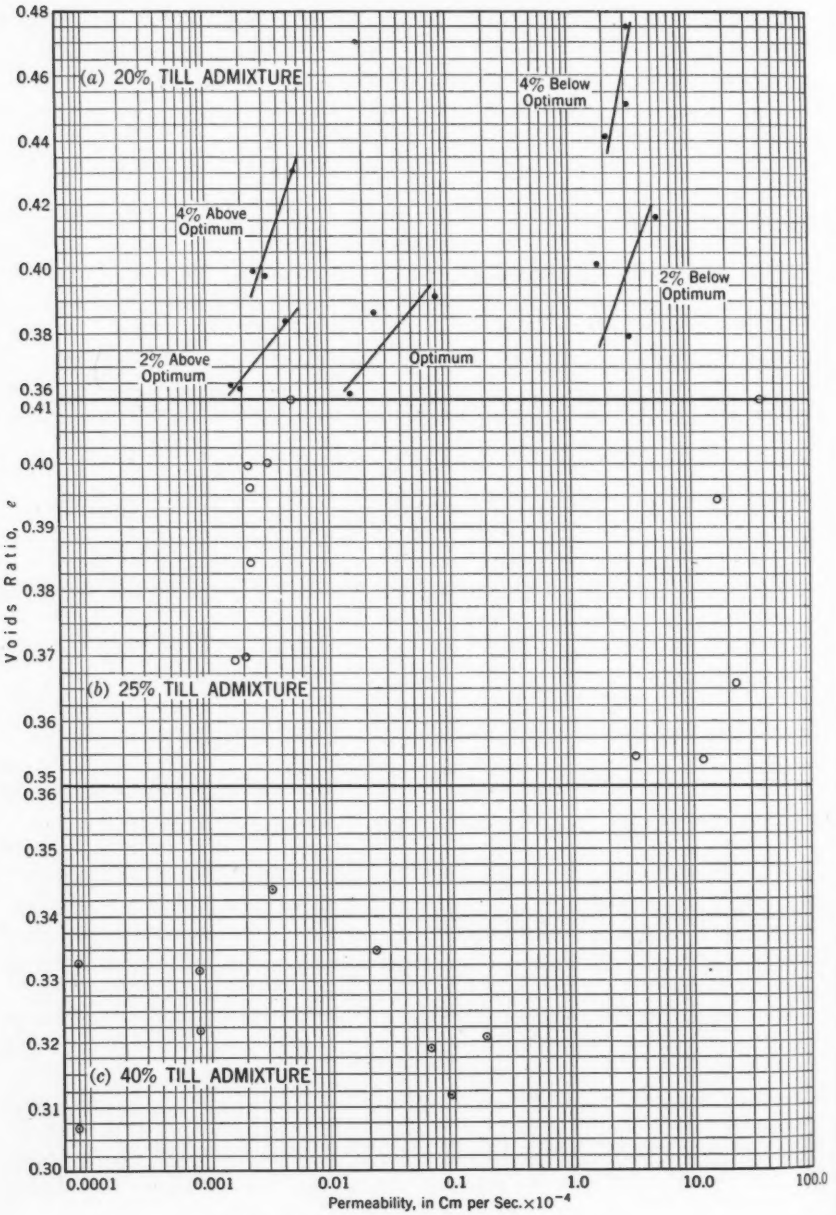


FIG. 4.—PERMEABILITY VERSUS VOIDS RATIO

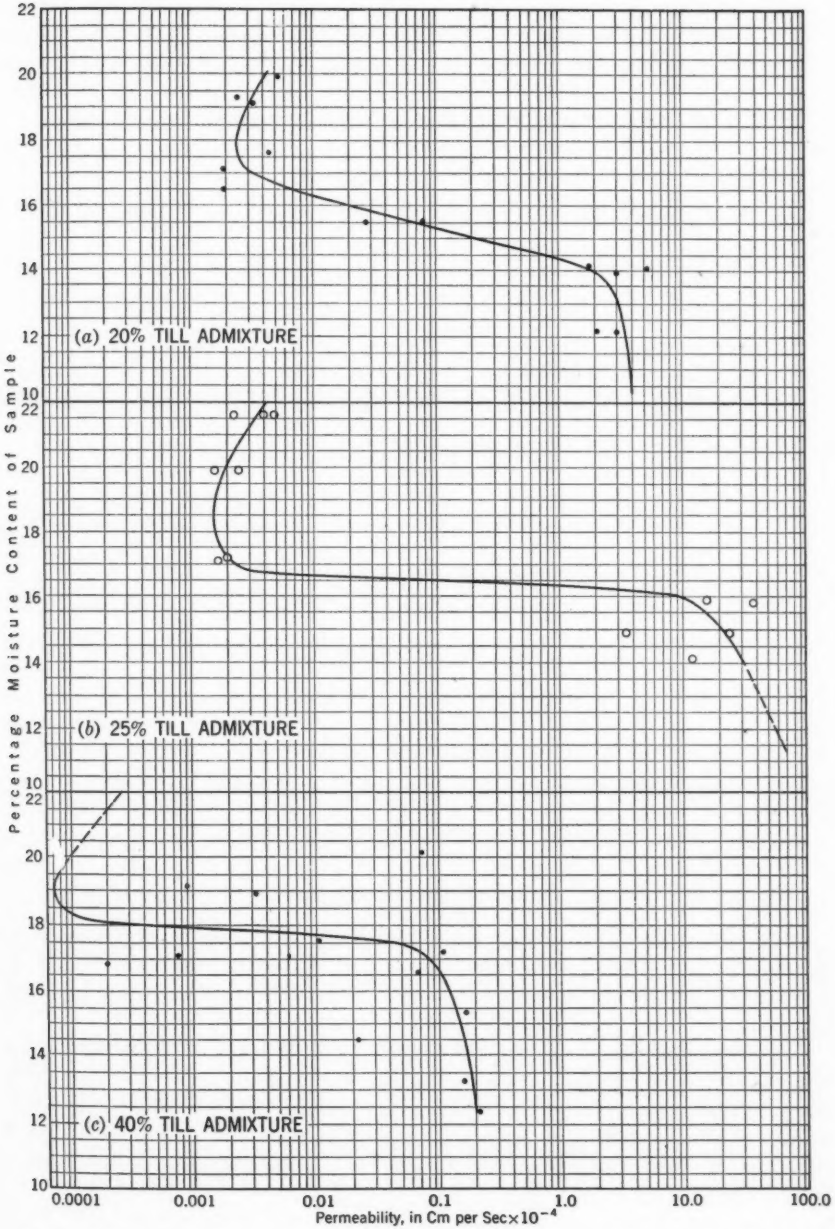


FIG. 5.—PERMEABILITY VERSUS MOISTURE CONTENT OF SAMPLE WHEN PLACED IN PERMEAMETER

*Time Effects.*—In samples that are placed in the permeameters at a water content below the optimum, the permeability is high. When water is run through the sample for 56 days, the permeability may drop in certain cases from $100 \times 10^{-4}$ cm per sec to $50 \times 10^{-4}$ cm per sec. The greatest percentage of the reduction generally occurs during the first few minutes after water is

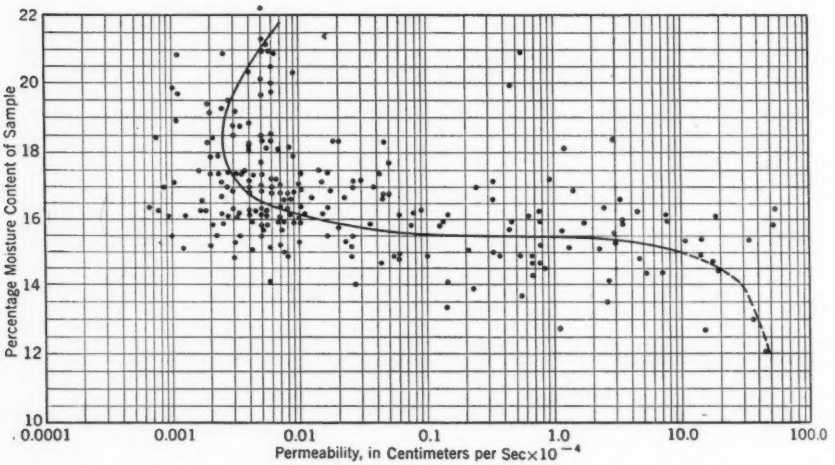


FIG. 6.—PERMEABILITY OF FILL MATERIAL VERSUS MOISTURE CONTENT OF SAMPLE WHEN PLACED IN PERMEAMETER

introduced into the sample. None of the individual samples have shown the extreme variations found in samples placed above and below the optimum. In samples placed above the optimum, the permeability remains low indefinitely, except for an occasional slight increase that may be attributed to removal of trapped air, which increases the effective voids.

## INTERPRETATIONS

When test results are so inconsistent with the usual approximation of permeability-void ratio relationship, the conclusion is inevitable (1) that there are large experimental errors in the tests or (2) that there are factors other than the numerical void ratio that may influence permeability.

After eliminating all experimental errors as far as possible without reducing the anomalies, it becomes evident that the void ratio expressed as a number does not define the nature of the voids. Numerically, the void ratio is a measure only of the total volume of voids. This total void volume may be distributed in an infinite variety of sizes and shapes of voids, depending on the distribution of soil particles.

In a granular soil of uniform particle size, the permeability-void ratio correlation is fairly consistent. This is explained by postulating that the size and distribution of the voids are more or less constant. In a well-graded soil with a small amount of clay included, consistent permeability-void ratio correlation can be obtained only when, and if, the particle and void distribution can be controlled. Control may be exercised under two conditions:

(1) In any given sample, the void ratio may be reduced progressively by consolidating the sample without appreciably disturbing the particle and void distribution. If permeability tests are run concurrently with the consolidation and the permeability-void ratio relationship is plotted on semilogarithmic paper, the plot will approximate a straight line.

(2) In a series of "identical" samples of the soil, the particle and void distribution appears to be controlled by exercising control of the water content at which the sample is compacted into the permeameter. At a constant water content but with varying void ratio, the results of permeability tests when plotted on semilogarithmic paper again approximate a straight-line permeability-void ratio relationship.

The optimum water content of a 1-cu-ft sample of Mud Mountain core material, with a maximum particle size of 3 in. as determined by a compaction investigation, is 11.5% of the dry weight of the entire sample. In the same sample, the water content of the fractions passing a ¼-in. screen is 16.5%. Now, if the optimum of a small sample using only those fractions passing a ¼-in. screen is determined, it is found to be 15.0%. Thus, when the 1-cu-ft sample containing the 3-in. maximum particle size is at its optimum water content, small aggregations of those fractions passing a ¼-in. screen are 1.5% above their optimum. With the fractions passing a ¼-in. screen slightly above their optimum, the clay, and particularly colloidal fractions, are in a very plastic condition and are relatively free to be uniformly distributed throughout the soil system during the compaction process. The montmorillonite clay becomes very unstable in the presence of water and lends itself readily to such a uniform distribution. Any increase in the water content at which the sample is compacted above this stage is reflected in a general increase in all voids in the soil, and the permeability-void ratio relationship then approaches a straight line (see Fig. 4), reflecting the numerical void ratio.

If this same sample were to be compacted into a cylinder at a water content of 1.5% below the optimum, the aggregations of finer particles will be at their optimum, will be well compacted, and, therefore, will have a relatively low void ratio. Surface tension and cohesion acting within the clay aggregations will be contributing factors in maintaining the aggregations at their maximum density, thereby enhancing a nonuniform particle and void distribution throughout the soil mass. The entire sample, being below its optimum, will have a relatively high void ratio because it cannot be compacted as densely as when at the optimum. Any decrease in the moisture content at which the sample is compacted below this stage is reflected in further decrease in the density and an increase in certain of the voids in the soil, and the permeability-void ratio relationship approaches a straight line (see Fig. 4), again reflecting the numerical void ratio.

The numerical void ratio appears to govern the permeability at the upper and lower limits of water content at which the materials were tested, but, at a relatively narrow range of water contents near the optimum, enormous changes of permeability take place with no change in void ratio necessary. This range of water content marks the transition from a condition of more or less uniform distribution of the fines on the wet side to a condition of concentration of fines into aggregations held together by cohesion and surface tension on the dry side. It is interesting to note that this range of moisture on either

side of the optimum is almost the exact range originally specified for field control on Mud Mountain Dam.

### Possible Application

In determining the permeability of certain pervious foundation materials, it is sometimes quite impossible to obtain undisturbed samples or to test the material in place, in which case it is common practice to determine the permeability of remolded samples over a range of void ratios. If this is done in conjunction with a single consolidation test, it has been suggested that the permeability-void ratio relationship should approach a straight line. However, if a series of tests is run on samples of the material compacted at various water contents, the results will indicate the degree of confidence that may be placed in the predictions of seepage through the foundation materials. If wide variations in permeability are obtained that cannot be attributed to void-ratio change, some other means of determining the permeability of the foundation materials is indicated.

The early practice of puddling cores in earth dams accomplished the same end as is attained by compacting this material at some moisture content slightly above the optimum. Puddling results in a uniform soil particle distribution, in a uniform void distribution, and hence in a low permeability for that particular soil. The fact that, in the Mud Mountain core material, there is only 1.5% to 3% of clay, and in a puddled core there may be nearly 100% of clay, does not alter the fact that a high water content assists in a uniform particle distribution.

### Further Investigations

The investigation has been limited thus far to a single material that is rather distinctive in its particle-size distribution. In this material, the effects of moisture are so extremely pronounced that they completely mask, over a certain range, all effects of numerical void-ratio change. Investigations of materials from four other sources, with a narrower range of particle sizes and no montmorillonite clay, are being conducted. It is indicated that the results will not reach the almost impossible extremes of the results obtained by using Mud Mountain borrow-pit material, but that herein may lie the answer to certain minor variations from the expected straight-line permeability versus void-ratio relationships.

### Authority and Acknowledgment

The laboratory investigation that forms the subject of this paper was conducted in the U. S. Engineer Soils Laboratory, at Mud Mountain Dam, under the direction of Capt. J. D. Lang, Frank R. Preble, William E. Hoy, and Albert H. Rodes, successive resident engineers on the Mud Mountain Dam project. The writers wish to express their appreciation to the entire laboratory personnel for their critical and painstaking work in developing the testing methods and equipment, and for the interpretations as herein set forth. The constant interest and friendly criticism of A. E. Jaskar, Jun. Am. Soc. C. E., and H. E. Johnson deserve special mention. H. R. Cedergren, Jun. Am. Soc. C. E., of the Portland (Ore.) Division, U. S. Engineers, and Profs. R. G. Hennes, Assoc. M. Am. Soc. C. E., and J. Hoover Mackin of the University of Washington, at Seattle, have read the manuscript and offered a number of valuable suggestions.

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PAPERS

# ENTRAINMENT OF AIR IN FLOWING WATER

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NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by February 1, 1942.

# OPEN CHANNEL FLOW AT HIGH VELOCITIES

By L. Standish Hall,[1] M. Am. Soc. C. E.


## Synopsis

One of the problems in the engineering design of chutes or steep spillway channels is the proper freeboard allowance to be made for the entrainment of air at high velocities. Calculations of the channel dimensions also must be modified with respect to the application of normal frictional resistance which is altered as the result of air entrainment. In all cases where water flowing in a steep chute is allowed to accelerate over a sufficient distance, air is mixed with the stream and the phenomenon of "white water" results. The same admixture of air with water can be observed in waterfalls or in natural stream channels where the velocity is sufficiently high to draw air into the water by its agitation.

Very few complete observations have been made for the measurement of the volume of air which will be mixed with the water under these conditions. Model tests are futile as sufficient velocity cannot be attained to create the phenomenon. To gather information as to the behavior of water flowing in steep channels and to measure the bulking of the water, some observations in actual spillway chutes were made in 1936 and 1937. The Pacific Gas and Electric Company granted permission for measurements to be made in three of their chutes and cooperated in maintaining uniform discharges during the period of the tests.

Incidental to the measurement of the bulking of the water, it was possible also to observe the performance of the stream while it flowed around vertical and horizontal curves in the spillway channels. The data gathered during these tests are presented in this paper, together with theoretical analyses leading to the development of a formula for computation of air bulking and general conclusions as to its application.


## Field Procedure

Ordinary field methods cannot be used in measuring velocities ranging from 35 to 80 ft per sec. The initial test was made on Hat Creek No. 2 chute in a western state. The spillway channel was laid out for observation on July 11, 1936, and measurements were taken on the following day. To obtain more complete data on this chute a second set of observations was made on April 17, 1937.

[1] Hydr. Engr., East Bay Municipal Utility Dist., Oakland, Calif.

The second test was made in Rapid Flume on July 20 and 21, 1936. The third test was made in the South Canal chute of another river project. The gages were located and levels run on April 22 to 24, 1937. The test was made on May 5.

The general field procedure was first to place gages along the chute channel for the purposes of measuring the velocity and cross-sectional area. The distances were measured along the actual slope of the chute rather than using the horizontal distances, stations being marked every 25 ft or 50 ft by scribing a paint mark and the station number on the side of the flume. For the concrete chutes, gages were stenciled on both inside surfaces of the channel at right angles to the bottom slope, using a prepared templet, the paint being a quick-drying black lacquer applied with a long-handled paint brush. Painted gages were not used in the wooden channel at Rapid Flume, but a measuring point was established on top of the flume and distances were measured down to the water surface. Levels were then run to each station and the elevation of the top of the chute wall recorded. At the same time the channel dimensions were checked.

The procedure for the actual observations, following this preliminary work, consisted of several steps: First, the discharge in the power canals above the chutes was gaged before and after the test. Four discharge measurements were usually made from a gaging bridge, using both the Price and Hoff current meters, two measurements being taken at the beginning of the test and two at the end. Observations were also taken at frequent intervals of a gage installed in the power canal to note any fluctuations in the water level. Then, after water had been turned into the chute and the flow had become steady, readings were taken on all gages along the channel. When the water is traveling at a high velocity with entrained air, particles of water leave the main body of the stream with the result that the water level is choppy and oscillates up and down. Therefore, observations were taken of the maximum and minimum levels at each station over a sufficient period of time to note the extreme range. A mean of these readings was then used as the average water level. If cross waves developed in the chute, due to curvature or entrance conditions, additional readings were taken across the channel by measuring down to the water surface from a board laid across the top of the chute.

Finally, velocity measurements were taken between various stations along the channel. Prior to making the measurements coloring matter was mixed, using either flourescein or potassium permanganate. Commercial flourescein was used, with lime first added to the mixing water to insure more complete solution of the crystals. About one teaspoonful of flourescein to a pint of water was used for each 240 cu ft per sec of discharge. For the potassium permanganate about ten teaspoonfuls to a pint of water were used for each 240 cu ft per sec of discharge. The colored solutions were poured into half-pint pasteboard containers prior to insertion in the stream channel. Before the test was begun, the observers were placed in their positions along the flume. Men with the dye solutions were stationed at the head of the flume or at the start of the course to be timed. A second observer with the revolver was located at the initial station, and the timers with stop watches were located

at the lower station. At a given signal, the container with the dye was dropped into the head of the channel and the upstream observer noted the passing of the color at his station by firing the revolver into the air. The timers at the lower end of the stretch started their watches on this signal and stopped them when the center of the slug of color passed the lower station. Four to six observers with stop watches were used to time each color cloud, and the average of their observations was used in determining the mean time interval.

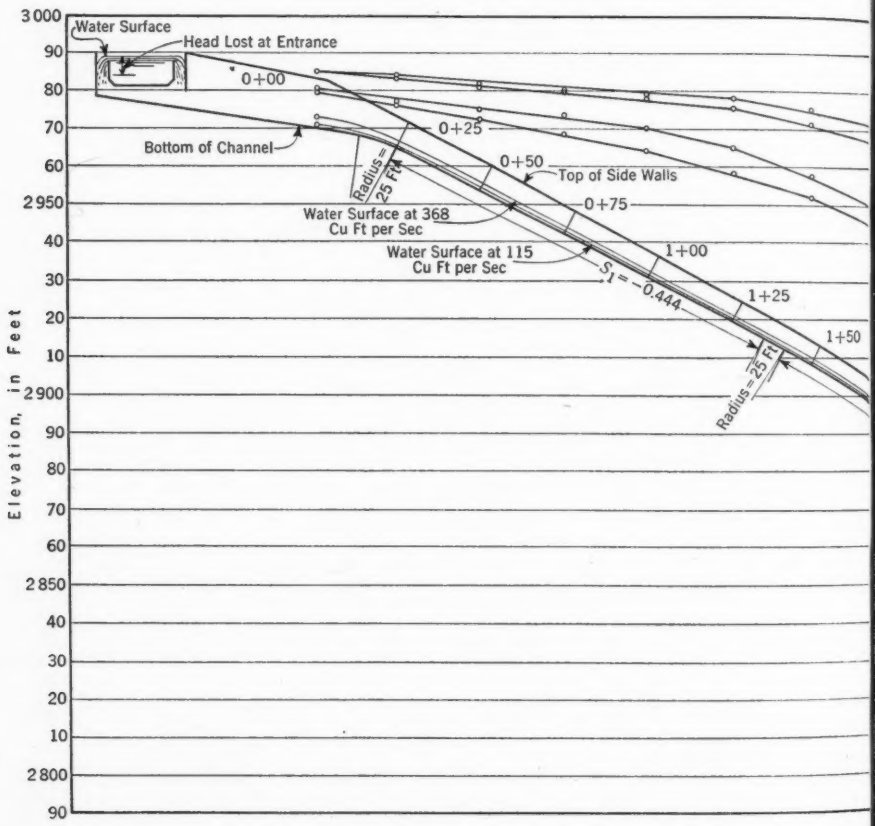


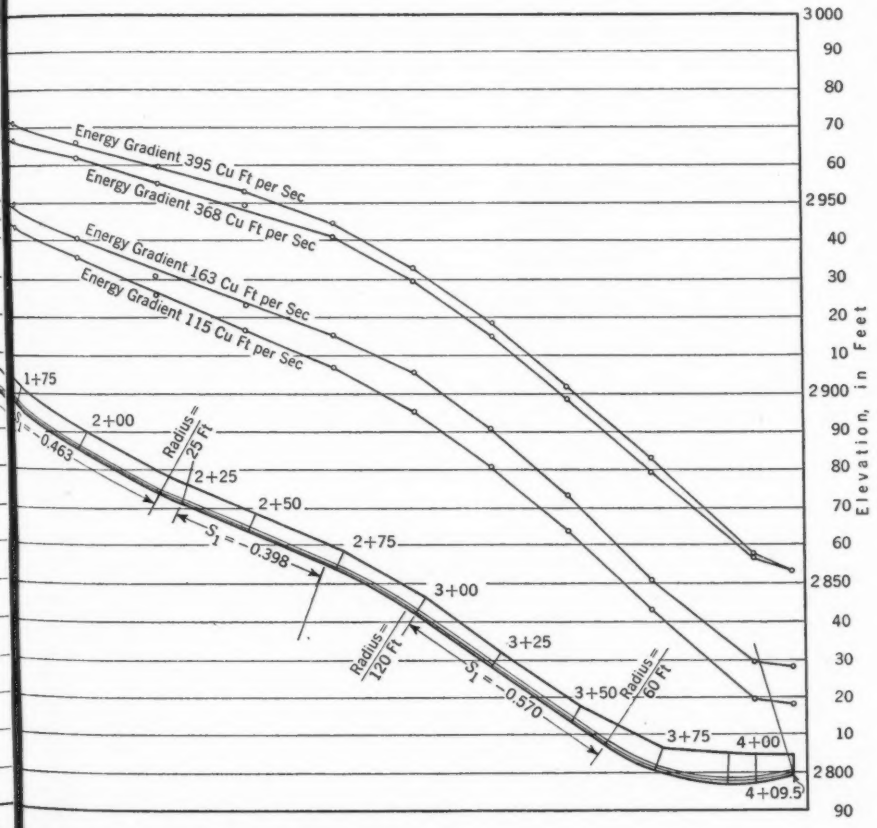

FIG. 1.—OBSERVED VELOCITIES, LONGITUDINAL WATER

## CONDITIONS DURING TEST

*Hat Creek Chute.*—The chute at the Hat Creek No. 2 power plant is located a short distance upstream from the penstock leading to the power house. This is a steep straight concrete channel having vertical curves and dropping 189 ft vertically in a distance of 409.5 ft measured along the channel. The waste water from the flume was discharged through side-channel spillway weirs located on both sides of the canal. After dropping over the weir lip the dis-

charge was gathered into four collecting channels which unite at the head of the chute.

The concrete of the chute was rather rough, with pronounced form marks. The tie wires used in constructing the forms had not been clipped off from the concrete and these added to the roughness of the surface. The channel had a uniform width of 5.75 ft with side-walls 5 ft high except near the entrance and outlet where the height was increased.




WATER SURFACE AND ENERGY PROFILES, HAT CREEK NO. 2 CHUTE

Due to two of the collecting channels entering the chute inlet at a right angle to the direction of flow, a cross wave was developed creating rather turbulent entrance conditions and also increasing the percentage of air in the water at this point. However, the cross wave and a portion of the entrained air had disappeared in about 100 ft. The profile of the chute is shown in Fig. 1. The first 275 ft of the chute consisted of tangents connected by short-radius vertical curves. Immediately below Station 2+75 there was a short convex vertical curve leading to a steep pitch having an inclination of 34° 45′. The

lower part of the curve below Station 3+60 was a concave vertical curve forming the outlet bucket.

The flow in the chute accelerated to Station 2+75. In this section of the channel running waves formed along the side-wall and gave the appearance of higher water levels than in the center of the channel. However, a check made by measuring the depth to water in the center portion of the stream showed that the maximum height reached by the water was the same as that observed on the side-walls. The upper surface of the stream was very turbulent, with balls of water that would dart up from the main body of the water surface at high velocities to drop back again in a short distance into the general mass of aerated water. The entire surface of the flow was composed of these small waves or rollers which ranged from 0.4 ft to 0.6 ft in height above the main body of the stream.

At Station 2+75 the curvature was too sharp to be followed by the trajectory of the flow, with the result that the upper layers of water left the bottom of the channel. At the maximum flows some of the spray overtopped the side-walls in this steep section of the chute. The cavitation caused some erosion of the concrete. The flow returned to the bottom of the chute between Stations 3+25 and 3+75, depending on the volume of flow. The velocity of the water was appreciably reduced in this section of the channel, where the stream was spread through a larger cross-sectional area. As a result, the water left the bucket at the lower end of the chute with a lower velocity than that observed at Station 2+75. The jet from the bucket discharged into the stream channel of Hat Creek above the normal stream water level, the energy being dissipated by eddies and turbulence in the pool formed in the channel by the discharge. The general layout at the chute and the flow conditions during the test are shown in Figs. 2, 3, and 4. A view of the flow conditions with a discharge of 395 cu ft per sec is presented in Fig. 2. At this station the velocity accelerates from 28 ft per sec to 55 ft per sec. At Station 2+70 water is apparently slightly higher along the sides of the chute than in the center due to wave action caused by retardation along the side-walls. The velocity at this point is 75 ft per sec.

### Rapid Flume

This chute is located just above the penstock leading to the high-head plant at a power house. The flume is used to discharge water from the Upper Standard Canal down the slope into a natural stream channel leading to Lake Tabeaud, where it is used afterward through the low-head plant. It is a wooden chute having a horizontal curve to the right near the upper end and a concave vertical curve near the terminal end of the channel. The total vertical drop is 145 ft in a distance of 667 ft measured along the chute. The discharge of water from the Upper Standard Canal into the flume is controlled by a head gate. Water passes from the canal through a culvert pipe and drops into the tapered inlet of the flume, creating a considerable initial turbulence in the water. In addition to this tapered section the chute channel is built on a

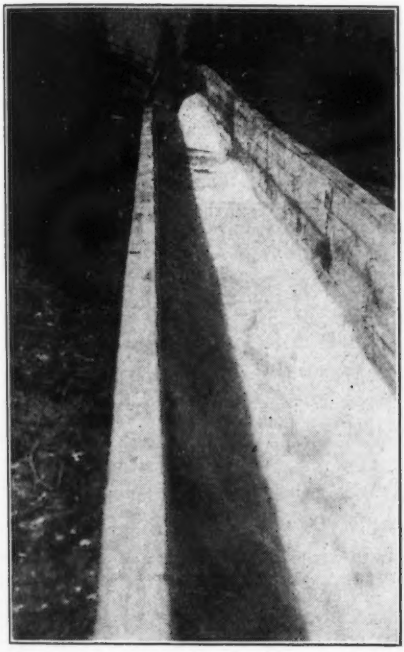


FIG. 2.—FACING UPSTREAM FROM STATION 1+25, HAT CREEK CHUTE; $Q$ = 395 CU FT PER SEC

FIG. 3.—WATER LEAVES THE BOTTOM OF THE CHUTE AT AN ABRUPT CHANGE OF GRADE (HAT CREEK CHUTE; $Q$ = 395 CU FT PER SEC)

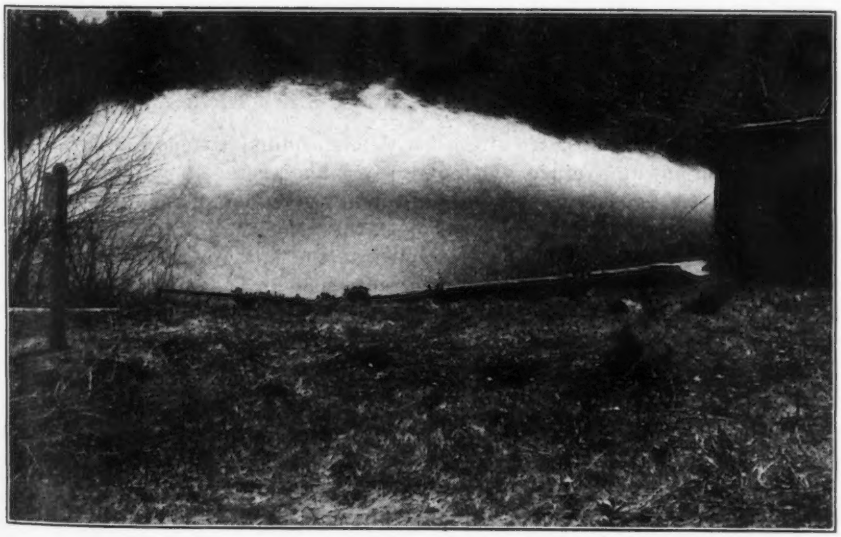


FIG. 4.—DISCHARGE OF 395 CU FT PER SEC FROM THE END OF HAT CREEK CHUTE; TERMINAL VELOCITY, 58 FT PER SEC

horizontal curve to the right to Station 2+00. A steep section of the chute, starting at Station 1+00 (Fig. 5), gradually merges into a long concave vertical curve of variable radius extending from Station 2+75 to Station 5+25. Below Station 5+25 the flume is built on a flatter grade down to the baffle box used to check the velocity. The water discharges from the baffle box into the natural drainage channel described previously.

FIG. 5.—FACING DOWNSTREAM ALONG RAPID FLUME AT STATION 1+100; VELOCITY, 35 FT PER SEC

The channel is constructed of wood with longitudinal battens, and below the taper section has an average width of 4.6 ft. The average height of the side-walls is 2.4 ft. Cross waves develop in the upper section of the flume, due to the horizontal curve. These disappear as the velocity is accelerated on the steep straight section. The air entrained at the inlet to the flume due to entrance conditions is partly lost and the entrained air is again drawn into the water, the insufflation commencing along the side-walls of the flume until the entire cross section below Station 1+50 has become aerated. Where turbulent entrance conditions do not exist the entrainment of air commences along the side-wall, as described in 1936 by E. W. Lane,[2] M. Am. Soc. C. E.

The velocity in Rapid Flume accelerated to a maximum of 52 ft per sec at Station 3+75 but gradually diminished as the grade flattened until it was reduced to a velocity of 33 ft per sec as the water passed into the baffle box. The profile of the flume is shown in Fig. 6.

[2] "Recent Studies of Flow Conditions in Steep Chutes," by E. W. Lane, *Engineering News-Record*, June 2, 1936, pp. 5–7.

### South Canal Chute

This chute is located at the end of the South Fork canal which picks up water below a power house. It is planned ultimately to use this water through Power Plant No. 6, but at the present time the flow is only used for irrigation. The water discharges down the chute into a natural channel. This channel is 3,535.25 ft long measured along the slope and has a total vertical drop of 140 ft. The discharge from the end of the South Canal is controlled by a radial gate set across the channel. Water can also discharge by a side channel spillway which drops the water back into the main chute channel within a distance of 100 ft below the radial gate. Water dropping into the chute from the side channel augmented the percentage of entrained air at the head of the chute. If no water had entered by this route, a much greater distance of channel would be traveled by the flow before the entire cross section was aerated.

The channel of the chute was constructed of concrete, the surface being rather rough and pitted, probably due to sand being carried down the chute at high velocities. On the bottom particularly, the cement and sand had been worn away, leaving the aggregate fragments exposed. From the head of the channel to Station 23+00 the width of the cross section was 10 ft. From Station 23+00 to Station 23+50 there was a tapered section and from the latter station to the end of the chute the channel had a width of 7 ft. The side-walls on the 10-ft section varied in height from 3.5 ft to 4.5 ft whereas on the 7-ft section the height of the side-walls varied from 3.5 ft to 5.25 ft.

There are thirteen horizontal curves of radii varying from 100 ft to 300 ft, the sharpest having an internal angle of 98° and a radius of 142 ft. The floor of the channel on these curves is not superelevated on the outer edge. An opportunity was afforded here to observe the flow of water around curves. The total angular deflection in the entire chute was 422° 32′. As stated, the entire length of channel was 3,535.25 ft, of which 1,328.8 ft was located on curves. There are numerous changes in grade, including three re-entrant grades inserted to decelerate the water at the toe of steep sections. Except for a steep pitch at the inlet, the steepest bottom slope is 0.100 for about 300 ft. The other principal grades are 700 ft at 0.055, 600 ft at 0.030, 200 ft at 0.022, 600 ft at 0.011, 715 ft at 0.025, and 170 ft in three short lengths at an average slope of 0.080.

A total time of 2 min 20 sec was required for a flow of 334 cu ft per sec to pass from the head of the chute to the outlet, the average velocity being about 25 ft per sec. Due to the many changes in grade, the velocity in various sections of the channel varied from 17 to 35 ft per sec. The profile of the channel is not shown, due to its great length.

As the channel bottom on the curves was not superelevated, the water piled up on the outside of each curve and produced standing cross waves in the tangent section below the curve (see Fig. 7), in addition to the choppy roller and surface waves elsewhere noted in normal chute flow. These cross waves lead to an error in computing the wetted area at some sections as only side gage

readings were taken. Such errors are not of sufficient magnitude to affect the results for the entire chute. In the 10-ft width of channel the superelevation of the water surface at the outer side of the curve did not form a uniform slope across the entire cross section, due to a larger percentage of the flow following the outer wall. However, on the 7-ft channel widths the slope of the water surface was more or less uniform so that the difference in level between the gage readings on the outer and inner walls of the curves indicated the slope of the water surface across the stream section. On the curve of 142-ft radius, in the 7-ft section, constructed on a 10% grade, the flow was very turbulent (see Fig. 8). The discharge in this section tended to proceed in a

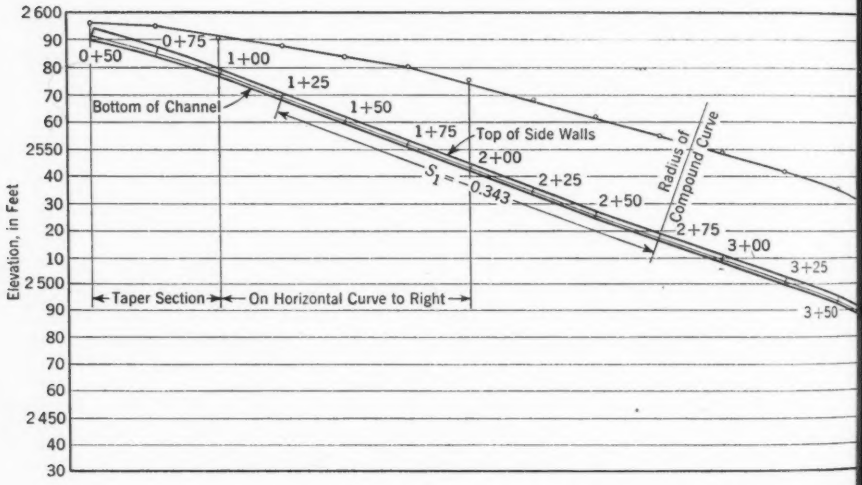



FIG. 6.—OBSERVED VELOCITY, LONGITUDINAL WATER SURFACE AND

series of chords with a result that water crowded at certain points on the outside of the curve, permitting spray to discharge over the top of the wall. Between these points the water surface was well below the top of the wall.

## NOTATION

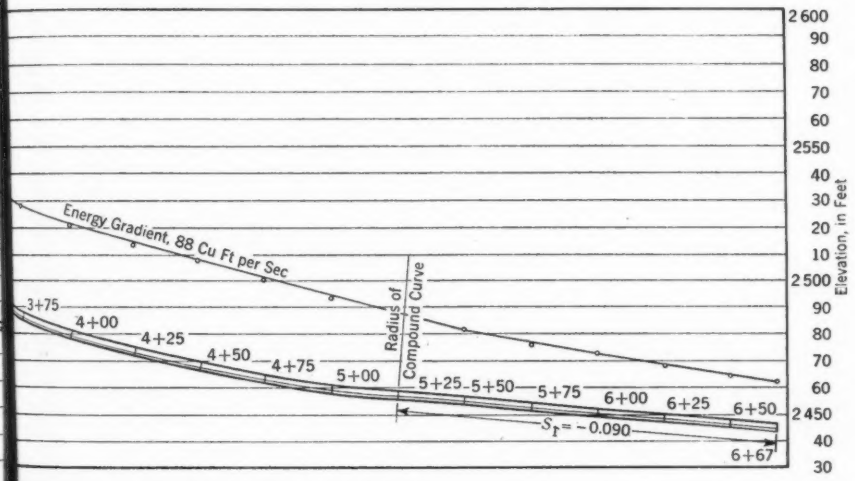
The letter symbols in this paper are defined where they first appear and are assembled for convenience of reference in the list of symbols appended to the Symposium.

## THEORY OF FLOW IN CHUTES

The objective, desired to be gained by the tests, was the determination of a formula for the calculation of air entrainment in projected spillway channels of large capacity. To be of practical application, the method of procedure should satisfy three requirements: (1) It should conform as nearly as practicable to established methods of hydraulic calculations; (2) it should be of sufficient simplicity to be readily applied; and (3) it should yield results that are in reasonable conformity with the measured tests of air entrainment.

Meager data available indicated that with small concrete chutes, having velocities ranging from 15 to 40 ft per sec, the entrained air varied from 15% to 35% of the cross section.[3] Measurements of a masonry lined chute in Austria gave air entrainment ranging from 60% to 80%.[4] The use of such large percentages for air bulking in design would make large spillway chutes for the discharge of river floods very costly.

It was indicated that the velocity, the distance traveled, and the roughness of the channel had some bearing on the air entrainment. Personal observation had led to the conclusion that air bulking was reduced with larger cross sections and greater volumes of discharge.




ERGY PROFILES, RAPID FLUME, TABEAUD RESERVOIR

In developing the theory of flow in chutes the following assumptions were made:

1. The value of the roughness coefficient $n$ in the Manning or Kutter formula is constant for the particular type of material of which the chute is constructed;

2. The air in and above the water caused no additional loss in energy, the reduction in the specific gravity compensating for the added area;

3. For computation, a computed hydraulic radius can be used

$$R_c = \frac{Q}{V P_c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

with a smaller value of $n$ than normal; and

[3] "The Determination of $n$ in Kutter's Formula for Various Canals, Flumes, and Chutes on the Boise Project and Vicinity," by W. G. Steward, U. S. Bureau of Reclamation, Rept. on 2d Annual Conference of Operating Men held at Boise, Idaho, January, 1913.

[4] "Overflow Channels from Surge Tanks," by Oberbauret Dr. Ing. Karl Innerebner, World Power Conference, 1924, Vol. II, p. 481.

4. The velocity head ($h_v$) computed from the mean velocity can be used without substantial error.

In Eq. 1, $Q$ is the discharge in cubic feet per second; $V$ is the observed average velocity in a section, in feet per second; and $P_c$ is the computed wetted perimeter of the section of water only, $A_c = \frac{Q}{V}$.

The true velocity head $h_v$ is greater than that computed from the mean velocity, since the true kinetic energy is the summation of the product of the velocity head and the weight of water for elemental areas, $a$, in the cross section. Then

$$\alpha = \frac{\int v^3 \, da}{V^3 A} \quad \ldots\ldots (2a)$$

and

$$h_v = \frac{\alpha V^2}{2g} \quad \ldots\ldots (2b)$$

in which $v$ is velocity through an element of area; $A$ is cross-section area of channel; and $g$ is acceleration due to gravity. The value of $h_v$ in Eq. 2*b* represents the average energy head per pound of water flowing per second; $\alpha$ is always greater than unity and ranges from 1.02 to 1.20. Since the velocity distribution in the chutes tested in this investigation was not obtained, it was assumed that the corrections were small and could be neglected. This assumption has the effect of slightly increasing the energy loss above the actual values. In the design of large spillway chutes, the inclusion of $\alpha$ is recommended and a value ranging from 1.07 to 1.10 is customary.

When air is entrained in water, the basic formula must be modified to

$$Q = \rho A V \quad \ldots\ldots (3)$$

in which $\rho$ = the ratio of water in a mixture of air and water or the relative density of the mixture ($1 - \rho$ = ratio of air). Assuming that the flow in a channel has reached a uniform velocity and neglecting internal friction of the water, the relation between the force represented by the weight of the water and the frictional resistance ($f$) of the channel walls for a cross-sectional area of elemental length can be expressed as $\rho A \, dl \, \gamma \sin \theta = f P \, dl$, or

$$f = \frac{\rho A \gamma \sin \theta}{P} \quad \ldots\ldots (4)$$

in which $l$ is the length measured along the channel; $\gamma$ is the weight of 1 cu ft of water (= 62.5 lb); $\theta$ is the slope of the channel in degrees; and $P$ is the wetted perimeter for a mixture of air and water.

When air is entrained in the water, the normal assumption given in most hydraulic textbooks as to the relation between the friction and the velocity for uniform flow must be modified as the entrained air reduces the density of the mixture and the friction correspondingly,[5]

$$f = k \rho V^2 \quad \ldots\ldots (5)$$

[5] "Applied Hydro- and Aero-Mechanics," by L. Prandtl and O. G. Tietjens, Engineering Societies Monographs, McGraw-Hill Book Co., Inc., 1934, p. 71.

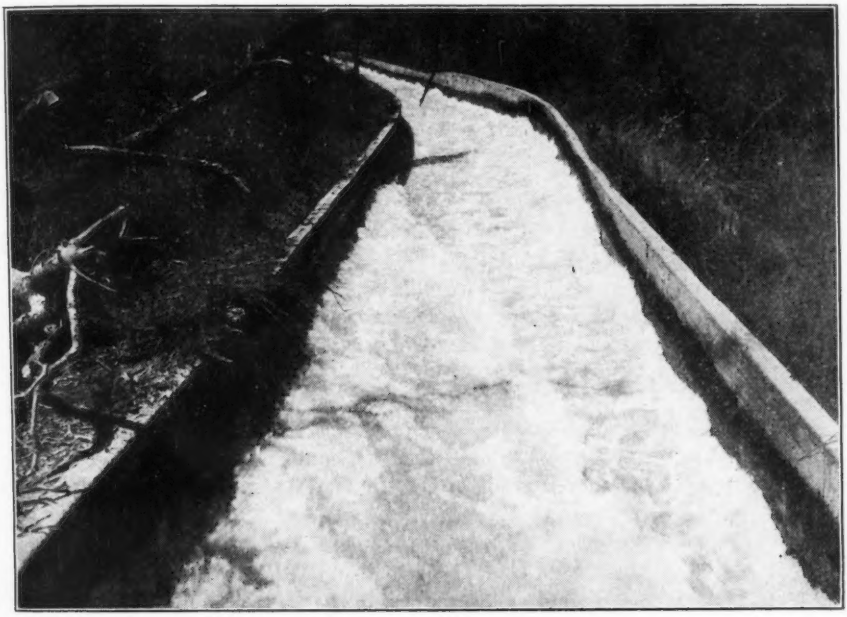


FIG. 7.—FACING UPSTREAM FROM STATION 7+75, SOUTH CANAL CHUTE

FIG. 8.—STATION 25+50, FACING UPSTREAM, SOUTH CANAL CHUTE; VELOCITY, 35 FT PER SEC

in which $k$ is a coefficient. The substitution of $R = \frac{A}{P}$ and $S_1 = \sin\theta$, the sine of slope of the channel, can be made in Eq. 4; then $k\,\rho\,V^2 = \rho\,R\,\gamma\,S_1$ and $V = \sqrt{\frac{\gamma}{k}\,R\,S_1}$ but $\sqrt{\frac{\gamma}{k}}$ = the Chézy $C$; hence

$$V = C\,\sqrt{R\,S_1} \qquad (6)$$

The observed hydraulic radius $R_o$ computed from the observed area $A_o$ and wetted perimeter $P_o$ of the mixture of air and water must be used. Then, the normal Chézy formula will apply.

One difficulty of applying this formula is that the ratio of air in the water $(1 - \rho)$ must be known in advance. This ratio is not constant but varies with the velocity, size of channel, and roughness of the side-walls. Hence, it becomes simpler to substitute for $R_o$ the value $R_c$, $A_c = \frac{Q}{V}$ and $R_c = \frac{A_c}{P_c}$.

To find the retardation factors for nonuniform flow with the data available from the chute tests, the values of Kutter's $n$ could be most easily computed from the slope of the energy gradient $S$ using the Manning formula as

$$S = \frac{n^2\,V^2{}_o}{2.2082\,(R_o)^{4/3}} \qquad (7)$$

If desired, the corresponding values of $n$ in the Kutter formula then can be determined by the relation between Manning's $n$ and Kutter's $n$ for various values of $R$.

If $R_c$ is used instead of $R_o$, a different value of $n$ must be used in Eq. 7 than the nominal value. The relation between the values of $n$ for the Manning formula would be

$$n_c = n\left(\frac{R_c}{R_o}\right)^{2/3} \qquad (8)$$

Since $R_c$ is always less than $R_o$, $n_c$ must be smaller than $n$. Values of $n_c$, therefore, were also calculated. Some writers have recommended the use of the observed area $A_o$ with a smaller velocity, $V_c = \frac{Q}{A_o}$. This procedure would necessitate the use of a larger value, $n_o$, which likewise has been computed from the available data. This method has the advantage of utilizing the actual area, but there are several objections to the use of a fictitious velocity less than the actual which will be pointed out later. The most logical procedure would appear to be the calculation of the actual velocity from the calculated depth without air and the determination of the true depth of air plus water from the percentage of entrained air. The relation between the entrained air and the velocity, hydraulic radius, and channel roughness is determined from experimental data.

Using Eq. 7, the computed values of $n$ are about the same as those in channels of similar material with velocities below critical. By using the observed velocity $V_o$, the corresponding slope of the energy gradient, $S_o$, and the computed hydraulic radius $R_c$, a smaller value of $n$ can be computed desig-

nated here as $n_c$. On the other hand, by using the observed hydraulic radius $R_o$, the computed velocity $V_c$ and the corresponding slope of the energy gradient $S_c$, a larger value of $n$ can be computed, designated here as $n_o$:

$$n_c = 1.486 \frac{(R_c)^{2/3} (S_o)^{1/2}}{V_o} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9a)$$

and

$$n_o = 1.486 \frac{(R_o)^{2/3} (S_c)^{1/2}}{V_c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9b)$$

The increment in velocity as flow progresses down the chute may be expressed as a function of the time. However, this leads to an equation involving hyperbolic functions that are awkward to use.[6]

A simpler solution can be derived by expressing the increment in velocity as a function of the distance traveled. Such an equation will be developed applying to a rectangular section of uniform width, $b$. Since many chutes are of this type, this formula will have wide application.

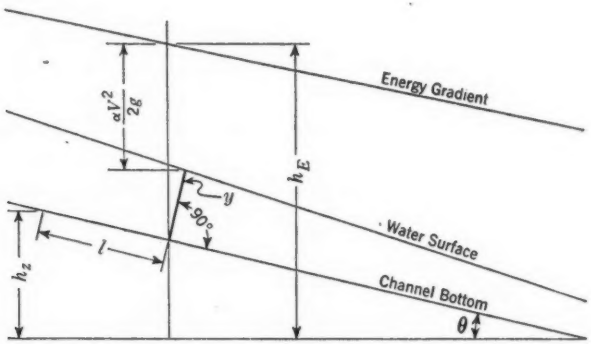



FIG. 9

Applying Bernoulli's theorem to Fig. 9,

$$h_E = h_z - l \sin \theta + y \cos \theta + \frac{\alpha V^2}{2g} \quad \ldots\ldots\ldots\ldots\ldots (10)$$

$\alpha$ being the velocity head coefficient from Eq. 2*b*.

Taking the derivative of $h_E$ with respect to $l$,

$$\frac{dh_E}{dl} = -\sin \theta + \cos \theta \frac{dy}{dl} + \frac{\alpha V}{g} \frac{dV}{dl} \quad \ldots\ldots\ldots\ldots\ldots (11)$$

In a rectangular channel, by substituting the computed depth $y_c = \rho y$ for the value of $y$, the discharge per foot of width $q$ becomes $q = V y_c$; but

$$dy_c = -\frac{y_c dV}{V} = -\frac{q dV}{V^2} \quad \ldots\ldots\ldots\ldots\ldots (12)$$

[6] "The Flow of Water in Channels Under Steep Gradients," by W. F. Durand, *Transactions*, A.S.M.E., 1940, p. 9 (presented at Semi-Annual Meeting of A.S.M.E., San Francisco, Calif., July 10–15, 1939).

Substituting Eq. 12 in Eq. 11,

$$\frac{dh_E}{dl} = -\sin\theta - \frac{\cos\theta\, q}{V^2}\frac{dV}{dl} + \frac{\alpha\, V}{g}\frac{dV}{dl} \quad \ldots\ldots (13)$$

but $S = \frac{dh_E}{dl}$, and substituting the value of $S$ as derived from Eq. 9*a*,

$$\frac{dh_E}{dl} = -\frac{n_c^2\, V^2}{2.2082\,(R_c)^{4/3}} \quad \ldots\ldots (14)$$

The negative sign is used in Eq. 14 since $h_E$ decreases with an increase in $l$. Equating Eq. 13 and Eq. 14 and solving for $\frac{dl}{dV}$

$$\frac{dl}{dV} = \frac{\frac{\alpha\, V}{g} - \frac{q}{V^2}\cos\theta}{\sin\theta - \frac{n_c^2\, V^2}{2.2082\,(R_c)^{4/3}}} \quad \ldots\ldots (15)$$

Since $R_c$ is a function of $V$, this value can be substituted in Eq. 15

$$R_c = \frac{\frac{b\,q}{V}}{b + \frac{2\,q}{V}} = \frac{b\,q}{b\,V + 2\,q} = \frac{q}{V + \frac{2\,q}{b}} \quad \ldots\ldots (16a)$$

or letting $\frac{2\,q}{b} = B$,

$$R_c = \frac{q}{V + B} \quad \ldots\ldots (16b)$$

Then

$$\frac{dl}{dV} = \frac{\frac{\alpha\, V}{g} - \frac{q\cos\theta}{V^2}}{\sin\theta - \frac{n_c^2\, V^2}{2.2082}\left(\frac{V+B}{q}\right)^{4/3}} \quad \ldots\ldots (17)$$

This is the basic equation for flow in steep rectangular chutes as derived from the simplified relation $q = V\, y_c$ rather than the more general equation

$$q = \rho\, V\, y_o \quad \ldots\ldots (18)$$

If the latter is used, three variables are introduced making the solution more difficult. Then,

$$\rho\, y_o\, dV + y_o\, V\, d\rho + \rho\, V\, dy_o = 0 \quad \ldots\ldots (19a)$$

$$dy_o = -\left(\frac{\rho\, y_o\, dV + y_o\, V\, d\rho}{\rho\, V}\right) \quad \ldots\ldots (19b)$$

A solution cannot be made unless $\rho = f(V)$. Then $d\rho = f_1(V)\, dV$.

In Eq. 19*b*, substitute $y_o = \frac{q}{\rho\, V}$ and $\rho = f(V)$:

$$dy = -\left[\frac{\rho\, q\, dV}{(\rho\, V)^2} + \frac{q\, V\, d\rho}{(\rho\, V)^2}\right] = -\left\{\frac{f(V)\, q\, dV}{[f(V)]^2\, V^2} + \frac{q\, V\, f_1(V)\, dV}{[f(V)]^2\, V^2}\right\}$$

$$= -\frac{q}{[f(V)]^2\, V^2}[f(V) + V\, f_1(V)]\, dV \quad \ldots\ldots (20)$$

Substitution of Eq. 20 in Eq. 11 gives

$$\frac{dl}{dV} = \frac{\frac{\alpha V}{g} - \frac{q \cos\theta}{[f(V)]^2 V^2}[f(V) + V f_1(V)]}{\sin\theta - \frac{n^2 V^2}{2.2082 (R_o)^{4/3}}} \quad \ldots\ldots\ldots\ldots (21)$$

The use of Eq. 21 would be difficult.

Returning to the basic Eq. 17, the solution for $\frac{dl}{dV}$ using a constant value of $n_c$ is not exact, since, by Eq. 8, $n_c = n\left(\frac{R_c}{R_o}\right)^{2/3}$. Hence, it is seen that $n_c$ is actually a function of $\rho$.

To determine an exact value of $n$ for chute computations, several assumptions may be made. It may be assumed that the slope of the energy gradient may be separated into two components representing frictional resistance and air resistance using the computed hydraulic radius $R_c$.[7] This solution has a fictitious character since the air resistance is based on the area occupied presumably by the water and not by the air. It was, therefore, discarded.

Another solution could be made by assuming that $\rho$ is constant for a given value of $\theta$. (A relation in steep chutes between the slope and the entrained air varying with the hydraulic radius was determined by R. Ehrenberger in 1930.)[8] Then Eq. 19*a* becomes

$$\rho\, y\, dV + \rho\, V\, dy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

In this case, the actual hydraulic radius $R_o$ should be used in Eq. 16*a*. Then by similar steps as used in deriving Eq. 16*b*,

$$R_o = \frac{q}{\rho V + B} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

This solution is at best an approximation since $\rho$ is not constant for a given value of $\theta$ but varies with the velocity. In many steep chutes the velocity accelerates throughout the length of the channel.

The most satisfactory solution was obtained by using the Froude number. The air entrained in the water would appear to bear some relation to the Froude number, as air entrainment is essentially related to the kinetic energy of the stream. The proper application of the dimensionless parameter to hydraulic phenomena has previously been pointed out[9] by Boris A. Bakhmeteff, M. Am. Soc. C. E., and Arthur E. Matzke, Assoc. M. Am. Soc. C. E.

In the present case, the frictional resistance, in addition to the kinetic energy, has a definite bearing on the air entrainment. Hence, it is not incongruous to use the hydraulic radius $R_c$ as the longitudinal parameter.

After some experimenting with the data, it was found that the best relation with the Froude number existed by using the ratio of the two ratios, air to water, in the mixture, or $\frac{1-\rho}{\rho}$.

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[7] Unpublished report of U. S. Bureau of Reclamation.

[8] *Die Wasserwirtschaft*, Vol. 23, 1930, pp. 573 and 595.

[9] "The Hydraulic Jump in Terms of Dynamic Similarity," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), p. 635.

Experimental data were plotted, as will be shown subsequently in the paper, using $\frac{1-\rho}{\rho}$ as ordinates and $\frac{V^2}{g\,R_c}$ as abscissa, and the following general relation was found to exist:

$$\frac{1-\rho}{\rho} = K\frac{V^2}{g\,R_c} - K_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (24a)$$

in which $K$ is a coefficient and $K_1$ is a constant for air entrainment. With much of the data it was found that $K_1 = 0$, but with smooth entrance conditions $K_1$ has a finite value. By transposition $\rho$ can be determined, assuming $K_1 = 0$, as follows:

$$\rho = \frac{1}{1 + \frac{K\,V^2}{g\,R_c}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (24b)$$

or, substituting for $R_c$ its value from Eq. 16*b*,

$$\rho = \frac{1}{1 + \frac{K\,V^2}{g\,q}(V + B)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (24c)$$

Dividing Eq. 16*b* by Eq. 23

$$\frac{R_c}{R_o} = \frac{\rho\,V + B}{V + B} \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

Substituting the value of $\frac{R_c}{R_o}$ from Eq. 25 in Eq. 8 and replacing $\rho$ by its value from Eq. 24*c*,

$$n_c = n\left[\frac{1 + \frac{B}{q}\frac{K\,V^2}{g}}{1 + \frac{K\,V^2}{g}\left(\frac{V+B}{q}\right)}\right]^{2/3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

Eqs. 24*b* and 26 constitute basic formulas for the solution of problems of air entrainment in spillway chutes. The use of values of $n_c$, obtained from Eq. 26, in Eq. 17 would appear to afford the most rational theoretical solution for flow in rectangular chutes.

In the design of chutes having a trapezoidal section, or in fact any shape other than rectangular, and in transition sections of any form in which the width varies, Eq. 17 cannot be applied. Similar formulas can be developed, but they are too cumbersome for practical use. An easier solution is obtained by a trial-and-error method.

Starting with Bernoulli's theorem from Eq. 10, the loss of energy between two adjacent stations can be expressed using subscripts 1 and 2 to indicate upstream and downstream sections, respectively. Equating this to the slope of the energy gradient in the Manning formula, the following equation results:

$$\frac{n^2_c\,Q^2}{2.2082\,A^2_c\,(R_c)^{4/3}} = \left(\frac{y_{c1} - y_{c2}}{l}\right)\cos\theta + \sin\theta + \frac{\alpha}{2g\,l}(V^2_1 - V^2_2) \quad \ldots (27)$$

In using Eq. 27, a value of $y_{c2}$ is first assumed and the corresponding value of $V_2$ is calculated. The value of $R_c$ to be used in this computation is the mean of the hydraulic radii at the two sections. If the two sides of the equation do not balance with the assumed value of $y_{c2}$, another value must be tried. Usually after two incorrect assumptions, the true value can be arrived at by interpolation. Eq. 27 can be used for rectangular sections, if desired. Calculations are facilitated by first preparing a series of curves of the several functions of each section for varying depths. These must be prepared for each discharge to be calculated. With the discharge $Q$ fixed, values of $A_c$, $R_c$, $P_c$, and $V$ can be determined readily. The ratio of water ($\rho$) can be calculated from Eq. 24*a* with the values of $V$ and $R_c$ for specific depths $y_c$, using the values of $K$ and $K_1$ assumed for the design. From the ratio of water in the section and the computed area, the actual area of the cross section can be computed:

$$A_o = \frac{A_c}{\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28a)$$

For trapezoidal sections, the actual depth of air and water $y_o$ can be found by the formula

$$y_o = \frac{-b_1 + \sqrt{b^2_1 + \frac{4\,s\,Q}{\rho\,V}}}{2\,s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (28b)$$

in which $b_1$ = bottom width and $s$ = side slopes in trapezoidal sections. Values of $A_o$, $R_o$, and $P_o$ then can be readily determined. For use in Eq. 27 values of $n_c$ are calculated from Eq. 8. For practical use, a curve of the function $n_c$ can be plotted against the depth $y_c$. The curve can be marked off to indicate increments in the depth for changes in $n_c$ to the nearest 0.0005. These values are sufficiently accurate and avoid too frequent change in the value of $n_c$, thereby facilitating the computations.

Friction loss computations in the Manning formula can be made most readily by first preparing a curve of the function $1.486\,A_c\,(R_c)^{2/3}$ plotted against the depth $y_c$. Once a value of $y_{c2}$ is assumed the friction loss can be computed in the left-hand member of Eq. 27 from the values of $n_c$ and of $1.486\,A_c\,(R_c)^{2/3}$ interpolated from the curves.

The computations of flow down the incline must be started from the head of the chute. With smooth entrance conditions, the computations can start from the critical section. When the velocity factor $\alpha$ is used, the formula for critical depth usually given in hydraulic texts must be modified by the introduction of this term:[10]

$$Q\sqrt{\frac{\alpha}{g}} = A_c\sqrt{\frac{A_c}{b_c}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

in which $b_c$ = water surface width in trapezoidal sections.

An advantage of determining the calculated areas of solid water is realized in the design of jump basin, which in many cases forms the chute terminal.

[10] Compare with formula for critical depth in "Hydraulics of Open Channels," by Boris A. Bakhmeteff, Engineering Societies Monographs, McGraw-Hill Book Co., Inc., 1932, p. 35.

Here, the values of $y_c$ and $A_c$ together with the actual water velocity $V_o$ at the toe of the chute should be used in the energy and momentum equations of the hydraulic jump. If the velocity factor $\alpha$ has been used in the energy equation, a similar factor based on the summation of $V^2$ over the cross section

TABLE 1.—GAGE READINGS, HAT CREEK NO. 2 CHUTE

| Station | South Side | | North Side | | Mean | South Side | | North Side | | Mean |
|---|---|---|---|---|---|---|---|---|---|---|
| | High (1) | Low (2) | High (3) | Low (4) | (5) | High (1) | Low (2) | High (3) | Low (4) | (5) |
| Velocity | (a) 115 Cu Ft per Sec | | | | | (b) 163 Cu Ft per Sec | | | | |
| 0+00 | 1.2 | 1.0 | 2.0 | 0.6 | 1.20 | 2.0 | 1.6 | 3.2 | 0.6 | 1.85 |
| 0+25 | 1.7 | 1.5 | 1.6 | 1.0 | 1.45 | 2.2 | 1.8 | 2.0 | 1.4 | 1.85 |
| 0+50 | 1.4 | 1.1 | 0.6 | 0.4 | 0.88 | 1.3 | 1.0 | 1.0 | 0.6 | 0.97 |
| 0+75 | 0.9 | 0.6 | 1.2 | 0.9 | 0.90 | 1.1 | 0.9 | 1.4 | 1.1 | 1.13 |
| 1+00 | 0.7 | 0.4 | 1.0 | 0.8 | 0.72 | 1.0 | 0.7 | 1.2 | 0.9 | 0.95 |
| 1+25 | 0.8 | 0.6 | 0.9 | 0.7 | 0.75 | 1.1 | 0.8 | 1.2 | 0.9 | 1.00 |
| 1+50 | 1.0 | 0.8 | 0.9 | 0.6 | 0.82 | 1.1 | 0.8 | 1.1 | 0.8 | 0.95 |
| 1+75 | 1.2 | 0.9 | 0.9 | 0.7 | 0.92 | 1.1 | 0.8 | 1.3 | 1.0 | 1.05 |
| 2+00 | 1.1 | 0.8 | 1.0 | 0.8 | 0.92 | 1.0 | 0.8 | 1.3 | 0.9 | 1.00 |
| 2+25 | 0.9 | 0.7 | 1.1 | 0.9 | 0.90 | 1.1 | 0.9 | 1.2 | 0.9 | 1.02 |
| 2+50 | 0.9 | 0.7 | 1.1 | 0.9 | 0.90 | 1.1 | 0.9 | 1.4 | 1.0 | 1.10 |
| 2+75 | 1.0 | 0.8 | 1.0 | 0.7 | 0.88 | 1.1 | 0.9 | 1.4 | 1.0 | 1.10 |
| 3+00[a] | ... | ... | 1.8 | 0.8 | 1.55 | 3.3 | 1.2 | 3.4 | 1.3 | 2.30 |
| 3+25[a] | ... | ... | 1.1 | 0.8 | 1.08 | 2.9 | 1.0 | 3.2 | 1.2 | 2.08 |
| 3+50[a] | 1.2 | 0.8 | 1.2 | 0.8 | 1.00 | 2.3 | 0.8 | 2.0 | 1.3 | 1.60 |
| 3+75 | ... | ... | 1.1 | 0.9 | 0.85 | 1.2 | 0.8 | 1.5 | 1.1 | 1.15 |
| 4+00 | ... | ... | 1.0 | 0.8 | 0.90 | 1.4 | 1.0 | 1.8 | 1.4 | 1.40 |
| 4+09.5 | 1.2 | 0.8 | 1.0 | 0.8 | 0.95 | 1.8 | 1.3 | 1.6 | 1.0 | 1.42 |
| Velocity | (c) 368 Cu Ft per Sec | | | | | (d) 395 Cu Ft per Sec | | | | |
| 0+00 | 4.4 | 3.8 | 4.0 | 3.2 | 3.85 | 4.5 | 3.9 | 4.0 | 3.6 | 4.00 |
| 0+25 | 3.2 | 2.6 | 3.6 | 2.6 | 3.00 | 3.3 | 2.7 | 3.8 | 2.8 | 3.15 |
| 0+50 | 2.8 | 2.0 | 1.9 | 1.5 | 2.05 | 3.2 | 2.4 | 2.0 | 1.8 | 2.35 |
| 0+75 | 1.9 | 1.6 | 2.1 | 1.7 | 1.82 | 2.1 | 1.7 | 2.3 | 1.9 | 2.00 |
| 1+00 | 1.8 | 1.4 | 2.0 | 1.5 | 1.68 | 2.0 | 1.6 | 2.1 | 1.7 | 1.85 |
| 1+25 | 1.9 | 1.6 | 1.9 | 1.3 | 1.68 | 2.1 | 1.7 | 1.9 | 1.5 | 1.80 |
| 1+50 | 1.8 | 1.5 | 1.8 | 1.2 | 1.58 | 2.1 | 1.5 | 2.0 | 1.5 | 1.77 |
| 1+75 | 1.9 | 1.4 | 1.9 | 1.5 | 1.68 | 2.1 | 1.5 | 2.2 | 1.6 | 1.85 |
| 2+00 | 1.8 | 1.4 | 1.8 | 1.4 | 1.60 | 2.1 | 1.5 | 2.1 | 1.5 | 1.80 |
| 2+25 | 1.8 | 1.5 | 1.9 | 1.4 | 1.65 | 2.1 | 1.5 | 2.1 | 1.5 | 1.80 |
| 2+50 | 1.9 | 1.7 | 2.0 | 1.5 | 1.78 | 2.2 | 1.6 | 2.2 | 1.6 | 1.90 |
| 2+75 | 1.8 | 1.5 | 2.0 | 1.4 | 1.68 | 2.0 | 1.5 | 2.3 | 1.6 | 1.85 |
| 3+00[a] | ... | ... | 3.2 | 2.4 | 2.90 | 4.2 | 2.6 | 4.4 | 2.9 | 3.40 |
| 3+25[a] | ... | ... | 4.4 | 3.2 | 3.80 | 6.0 | 3.0 | 6.0 | 3.4 | 4.60 |
| 3+50[a] | ... | ... | 2.4 | 1.6 | 2.00 | 4.8 | 2.0 | 5.0 | 2.2 | 3.50 |
| 3+75 | 1.7 | 1.5 | 1.9 | 1.7 | 1.70 | 2.1 | 1.6 | 2.3 | 1.9 | 1.85 |
| 4+00 | 1.7 | 1.5 | 1.9 | 1.7 | 1.70 | 2.2 | 1.6 | 2.2 | 1.9 | 1.90 |
| 4+09.5 | 1.5 | 1.1 | 1.9 | 1.7 | 1.55 | 2.4 | 2.0 | 2.4 | 1.9 | 2.20 |

[a] Between these stations the flow did not follow the bottom of the flume because the vertical curve was sharper than the trajectory of the flowing water.

must be applied to the momentum equation. This factor is smaller than $\alpha$ and values ranging from 1.04 to 1.07 are generally applied. (The correction to the average velocity applicable to the energy and momentum equations has been determined by Coriolis, De Saint Venant, and others.)[11]

[11] See *Annales des Ponts et Chaussées*, 1836; also *Transactions*, A.S.M.E., Vol. 54, 1932, p. 57.

Calculation of the jump basin dimensions in this manner will avoid the inadequate lengths and depths encountered in many constructed designs.

Another of the problems in the design of chutes is the determination of the path of the water around vertical and horizontal curves. Concave vertical curves present no difficulty in design, except for stress created by the centrifugal force. Convex vertical curves must not be made sharper than the trajectory that would be followed by the high velocity water under the action of gravity. If the flume bottom is steeper than this trajectory, the water will leave the bottom and may rise above the top of the side-walls as observed at Hat Creek chute. The trajectory will be a parabola tangent to the slope of the chute.

TABLE 2.—CORRECTIONS TO OBSERVED TIMES ON FULL RUNS IN THE LOWER PART OF HAT CREEK NO. 2 CHUTE

| TEST DATE, JULY 12, 1936 | | | | TEST DATE, APRIL 17, 1937 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Full Run Stations | | Time (Sec) | | Full Run Stations | | Time (Sec) | | Partial Run Stations | | Time (Sec) | | Ratio, Col. 11 / Col. 12 |
| Upper | Lower | Observed[a] | Corrected | Upper | Lower | Observed[a] | Corrected | Upper | Lower | Observed[a] | Comparative[b] | |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) |
| 115 CU FT PER SEC | | | | 163 CU FT PER SEC | | | | 163 CU FT PER SEC | | | | |
| 0+25 | 1+25 | 2.48 | 2.48 | 0+25 | 1+25 | 2.27 | 2.27 | 1+25 | 2+25 | 1.91 | 2.02 | 95 |
| 0+25 | 2+25 | 4.65 | 4.52 | 0+25 | 2+25 | 4.29 | 4.17 | 1+25 | 2+75 | 2.70 | 2.865 | 94 |
| 0+25 | 2+75 | 5.52 | 5.32 | 0+25 | 2+75 | 5.135 | 4.95 | .... | ..... | ..... | ..... | .. |
| 0+25 | 3+75 | 7.60 | 7.07 | 0+25 | 3+75 | 6.963 | 6.48 | 2+75 | 4+09.5 | 2.065 | 2.695 | 77 |
| 0+25 | 4+09.5 | 8.58 | 7.87 | 0+25 | 4+09.5 | 7.83 | 7.19 | .... | ..... | ..... | ..... | .. |
| 368 CU FT PER SEC | | | | 395 CU FT PER SEC | | | | 395 CU FT PER SEC | | | | |
| 0+25 | 1+25 | 2.29 | 2.29 | 0+25 | 1+25 | 2.16 | 2.16 | 1+25 | 2+25 | 1.46 | 1.51 | 97 |
| 0+25 | 2+25 | 3.89 | 3.79 | 0+25 | 2+25 | 3.67 | 3.58 | 1+25 | 2+75 | 2.24 | 2.58 | 87 |
| 0+25 | 2+75 | 4.50 | 4.35 | 0+25 | 2+75 | 4.74 | 4.56 | .... | ..... | ..... | ..... | .. |
| 0+25 | 3+75 | 6.05 | 5.65 | 0+25 | 3+75 | 6.22 | 5.80 | 2+75 | 4+09.5 | 1.74 | 1.96 | 89 |
| 0+25 | 4+09.5 | 6.68 | 6.17 | 0+25 | 4+09.5 | 6.70 | 6.19 | .... | ..... | ..... | ..... | .. |

[a] Observed times have been corrected by factors determined for length of travel.
[b] Equivalent time on full run.

The equation of the curve, neglecting air resistance, is an adaptation of the usual equations of the path of a projectile given in many engineering texts.[12]

To have acquired a velocity equal to the vertical component, the water would have dropped a vertical distance

$$y_t = \frac{V^2 \sin^2 \theta}{2g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30a)$$

In this same time the flow would cover a horizontal distance

$$x_t = \frac{V^2 \sin \theta \cos \theta}{g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30b)$$

[12] See *American Civil Engineers' Handbook*, 5th Ed., p. 119.

The point of tangency to the parabolic curve is at coordinates $y_t, x_t$ from the point of origin. The equation of the curve is $x^2 = m\,y$, in which

$$m = \frac{2\,V^2 \cos^2\theta}{g} \qquad (31)$$

In many constructed chutes, it is found that the water does not follow the bottom of the channel on convex vertical curves. This condition is probably

TABLE 3.—HYDRAULIC PROPERTIES,

| Station | Depths (Ft) | | Hydraulic Radius (Ft) | | Area (Sq Ft) | | Percentages | | Ratio, Col. 8 / Col. 7 | Observed velocity adjusted $V_o$ | Velocity head[a] $H_v$ | $\frac{V^2}{g\,R_c}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Observed $y_o$ | Computed $y_c$ | Observed $R_o$ | Computed $R_c$ | Observed $A_o$ | Computed $A_c$ | Water ($\rho$) | Air ($1-\rho$) | | | | |
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) |
| | | | | | (*a*) 115 Cu Ft per Sec | | | | | | | |
| 0+00 | 1.20 | 0.92 | 0.88 | 0.71 | 6.46 | 4.90 | 75.7 | 24.3 | 0.32 | 23.5 | 8.6 | 24.2 |
| 0+25 | 1.45 | 0.83 | 1.00 | 0.66 | 7.89 | 4.41 | 55.9 | 44.1 | 0.79 | 26.1 | 10.6 | 32.1 |
| 0+50 | 0.88 | 0.65 | 0.67 | 0.53 | 4.62 | 3.37 | 72.9 | 27.1 | 0.37 | 34.2 | 18.2 | 68.6 |
| 0+75 | 0.90 | 0.55 | 0.70 | 0.45 | 4.73 | 2.82 | 59.7 | 40.3 | 0.68 | 40.8 | 25.8 | 115.0 |
| 1+00 | 0.72 | 0.50 | 0.58 | 0.41 | 3.70 | 2.51 | 67.9 | 32.1 | 0.47 | 45.8 | 32.6 | 159.0 |
| 1+25 | 0.75 | 0.47 | 0.60 | 0.39 | 3.87 | 2.33 | 60.2 | 39.8 | 0.66 | 49.4 | 37.9 | 194.0 |
| 1+50 | 0.82 | 0.45 | 0.65 | 0.38 | 4.27 | 2.21 | 51.8 | 48.2 | 0.93 | 52.0 | 43.1 | 227.0 |
| 1+75 | 0.92 | 0.43 | 0.72 | 0.36 | 4.85 | 2.11 | 43.5 | 56.5 | 1.30 | 54.5 | 46.2 | 257.0 |
| 2+00 | 0.92 | 0.41 | 0.72 | 0.35 | 4.85 | 2.04 | 42.1 | 57.9 | 1.38 | 56.4 | 49.5 | 283.0 |
| 2+25 | 0.90 | 0.41 | 0.70 | 0.35 | 4.73 | 2.01 | 42.5 | 57.5 | 1.35 | 57.4 | 51.2 | 293.0 |
| 2+50 | 0.90 | 0.40 | 0.70 | 0.34 | 4.73 | 1.98 | 41.9 | 58.1 | 1.39 | 58.0 | 52.3 | 308.0 |
| 2+75 | 0.88 | 0.40 | 0.69 | 0.34 | 4.61 | 1.98 | 42.9 | 57.1 | 1.33 | 58.1 | 52.5 | 308.0 |
| 3+00 | 1.55 | 0.40 | 1.05 | 0.34 | 8.47 | 1.98 | 23.4 | 76.6 | 3.27 | 58.1 | 52.5 | 308.0 |
| 3+25 | 1.08 | 0.40 | 0.81 | 0.34 | 5.77 | 1.98 | 34.3 | 65.7 | 1.92 | 58.1 | 52.5 | 308.0 |
| 3+50 | 1.00 | 0.41 | 0.76 | 0.35 | 5.30 | 2.03 | 38.3 | 61.7 | 1.61 | 56.7 | 50.0 | 285.0 |
| 3+75 | 0.85 | 0.45 | 0.67 | 0.38 | 4.44 | 2.22 | 50.0 | 50.0 | 1.00 | 51.8 | 41.6 | 219.0 |
| 4+00 | 0.90 | 0.61 | 0.70 | 0.49 | 4.73 | 3.12 | 66.0 | 34.0 | 0.52 | 36.9 | 21.2 | 87.0 |
| 4+09.5 | 0.95 | 0.64 | 0.73 | 0.54 | 5.02 | 3.35 | 66.8 | 33.2 | 0.50 | 34.3 | 18.3 | 68.0 |
| | | | | | (*b*) 163 Cu Ft per Sec | | | | | | | |
| 0+00 | 1.85 | 1.23 | 1.18 | 0.89 | 10.19 | 6.65 | 65.3 | 34.7 | 0.53 | 24.5 | 9.3 | 20.8 |
| 0+25 | 1.85 | 1.13 | 1.18 | 0.84 | 10.19 | 6.06 | 59.6 | 40.4 | 0.68 | 26.9 | 11.2 | 26.7 |
| 0+50 | 0.97 | 0.85 | 0.75 | 0.67 | 5.13 | 4.49 | 87.5 | 12.5 | 0.14 | 36.3 | 20.5 | 61.2 |
| 0+75 | 1.13 | 0.71 | 0.85 | 0.57 | 6.05 | 3.68 | 60.8 | 39.2 | 0.64 | 44.4 | 30.7 | 108.0 |
| 1+00 | 0.95 | 0.63 | 0.74 | 0.52 | 5.02 | 3.28 | 65.4 | 34.6 | 0.53 | 49.7 | 38.4 | 148.0 |
| 1+25 | 1.00 | 0.60 | 0.77 | 0.49 | 5.31 | 3.05 | 57.4 | 42.6 | 0.74 | 53.5 | 44.6 | 182.0 |
| 1+50 | 0.95 | 0.57 | 0.74 | 0.48 | 5.02 | 2.92 | 58.2 | 41.8 | 0.72 | 55.9 | 48.7 | 203.0 |
| 1+75 | 1.05 | 0.55 | 0.80 | 0.45 | 5.59 | 2.82 | 50.5 | 49.5 | 0.98 | 57.7 | 51.7 | 230.0 |
| 2+00 | 1.00 | 0.54 | 0.77 | 0.45 | 5.31 | 2.75 | 51.8 | 48.2 | 0.93 | 59.2 | 54.5 | 242.0 |
| 2+25 | 1.02 | 0.53 | 0.78 | 0.45 | 5.42 | 2.70 | 49.9 | 50.1 | 1.00 | 60.4 | 56.7 | 252.0 |
| 2+50 | 1.10 | 0.52 | 0.83 | 0.44 | 5.88 | 2.66 | 45.3 | 54.7 | 1.26 | 61.4 | 58.6 | 266.0 |
| 2+75 | 1.10 | 0.52 | 0.83 | 0.44 | 5.88 | 2.61 | 44.4 | 55.6 | 1.25 | 62.4 | 60.5 | 275.0 |
| 3+00 | 2.30 | 0.51 | 1.34 | 0.43 | 12.78 | 2.57 | 20.1 | 79.9 | 3.98 | 63.5 | 62.7 | 292.0 |
| 3+25 | 2.08 | 0.51 | 1.26 | 0.43 | 11.52 | 2.58 | 22.4 | 77.6 | 3.46 | 63.3 | 62.3 | 290.0 |
| 3+50 | 1.60 | 0.52 | 1.08 | 0.44 | 8.76 | 2.64 | 30.1 | 69.9 | 2.32 | 61.7 | 59.2 | 269.0 |
| 3+75 | 1.15 | 0.57 | 0.86 | 0.47 | 6.17 | 2.89 | 46.8 | 53.2 | 1.14 | 56.5 | 49.7 | 211.0 |
| 4+00 | 1.40 | 0.70 | 0.98 | 0.57 | 7.60 | 3.64 | 47.9 | 52.1 | 1.09 | 44.7 | 31.1 | 109.0 |
| 4+09.5 | 1.42 | 0.70 | 0.99 | 0.59 | 7.72 | 3.86 | 50.0 | 50.0 | 1.00 | 42.3 | 27.8 | 94.2 |

[a] Velocity head, $H_v$,

due to the fact that the method of calculation did not yield the true velocity of the stream. Even if the true mean velocity has been determined by the method previously outlined, it is necessary to determine the maximum velocity in the channel which will occur near the surface of the center portion of the

cross section. In short chutes this maximum velocity will be practically equal to the theoretical velocity resulting from the existing vertical drop. In longer chutes, the maximum velocity will be from 15% to 20% greater than the mean velocity.[13] Professor Lane found that the maximum velocity in the Uncompahgre chute was 15% greater than the mean.[14]

Aside from the possibility of damage to the concrete in the chute from cavitation, or from the tendency to lift the entire bottom slab, there appears to

HAT CREEK NO. 2 CHUTE

| Station | Depths (Ft) | | Hydraulic Radius (Ft) | | Area (Sq Ft) | | Percentages | | Ratio, Col. 8 / Col. 7 | Observed velocity adjusted $V_o$ | Velocity head[a] $H_v$ | $\frac{V^2}{g R_e}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Observed $y_o$ | Computed $y_c$ | Observed $R_o$ | Computed $R_c$ | Observed $A_o$ | Computed $A_c$ | Water ($\rho$) | Air ($1-\rho$) | | | | |
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) |
| | | | | | (*c*) 368 Cu Ft per Sec | | | | | | | |
| 0+00 | 3.85 | 2.32 | 1.71 | 1.34 | 21.70 | 12.90 | 59.4 | 40.6 | 0.68 | 28.5 | 12.6 | 18.8 |
| 0+25 | 3.00 | 1.99 | 1.53 | 1.23 | 16.80 | 11.02 | 65.6 | 34.4 | 0.52 | 33.4 | 17.3 | 28.1 |
| 0+50 | 2.05 | 1.63 | 1.25 | 1.08 | 11.34 | 9.03 | 79.5 | 20.5 | 0.26 | 40.8 | 25.9 | 48.0 |
| 0+75 | 1.82 | 1.40 | 1.20 | 0.98 | 10.01 | 7.65 | 76.5 | 23.5 | 0.31 | 48.1 | 35.9 | 73.2 |
| 1+00 | 1.68 | 1.25 | 1.11 | 0.90 | 9.21 | 6.79 | 73.6 | 26.4 | 0.36 | 54.2 | 45.7 | 101.0 |
| 1+25 | 1.68 | 1.16 | 1.11 | 0.85 | 9.21 | 6.23 | 67.7 | 32.3 | 0.48 | 59.1 | 54.3 | 128.0 |
| 1+50 | 1.58 | 1.08 | 1.07 | 0.81 | 8.63 | 5.84 | 67.6 | 32.4 | 0.48 | 63.0 | 61.7 | 152.0 |
| 1+75 | 1.68 | 1.03 | 1.11 | 0.78 | 9.21 | 5.52 | 59.8 | 40.2 | 0.67 | 66.6 | 69.1 | 177.0 |
| 2+00 | 1.60 | 0.99 | 1.08 | 0.76 | 8.75 | 5.27 | 60.3 | 39.7 | 0.66 | 69.7 | 75.6 | 199.0 |
| 2+25 | 1.65 | 0.96 | 1.10 | 0.74 | 9.04 | 5.10 | 56.4 | 43.6 | 0.77 | 72.1 | 80.9 | 219.0 |
| 2+50 | 1.78 | 0.94 | 1.15 | 0.72 | 9.79 | 4.97 | 50.8 | 49.2 | 0.97 | 74.0 | 85.1 | 236.0 |
| 2+75 | 1.68 | 0.92 | 1.11 | 0.71 | 9.21 | 4.93 | 53.4 | 46.6 | 0.87 | 74.6 | 86.5 | 244.0 |
| 3+00 | 2.90 | 0.92 | 1.51 | 0.71 | 16.23 | 4.93 | 30.3 | 69.7 | 2.30 | 74.6 | 86.5 | 244.0 |
| 3+25 | 3.80 | 0.92 | 1.70 | 0.71 | 21.40 | 4.93 | 23.0 | 77.0 | 3.35 | 74.6 | 86.5 | 244.0 |
| 3+50 | 2.00 | 0.94 | 1.24 | 0.71 | 11.05 | 4.99 | 45.2 | 54.8 | 1.21 | 73.8 | 84.7 | 244.0 |
| 3+75 | 1.70 | 0.97 | 1.12 | 0.81 | 9.33 | 5.20 | 55.8 | 44.2 | 0.79 | 70.8 | 78.0 | 193.0 |
| 4+00 | 1.70 | 1.12 | 1.12 | 0.86 | 9.33 | 6.05 | 64.9 | 35.1 | 0.54 | 60.9 | 57.7 | 134.0 |
| 4+09.5 | 1.55 | 1.17 | 1.09 | 0.91 | 8.46 | 6.36 | 75.2 | 24.8 | 0.33 | 57.9 | 52.1 | 115.0 |
| | | | | | (*d*) 395 Cu Ft per Sec | | | | | | | |
| 0+00 | 4.00 | 2.48 | 1.74 | 1.39 | 22.56 | 13.84 | 61.5 | 38.5 | 0.63 | 28.5 | 12.6 | 18.1 |
| 0+25 | 3.15 | 2.14 | 1.57 | 1.28 | 17.67 | 11.83 | 67.0 | 33.0 | 0.49 | 33.4 | 17.4 | 27.2 |
| 0+50 | 2.35 | 1.75 | 1.36 | 1.13 | 13.07 | 9.67 | 74.0 | 26.0 | 0.35 | 40.8 | 25.9 | 45.8 |
| 0+75 | 2.00 | 1.50 | 1.23 | 1.02 | 11.06 | 8.20 | 74.1 | 25.9 | 0.35 | 48.1 | 36.0 | 70.6 |
| 1+00 | 1.85 | 1.33 | 1.18 | 0.95 | 10.19 | 7.21 | 71.0 | 29.0 | 0.41 | 54.7 | 46.6 | 98.1 |
| 1+25 | 1.80 | 1.20 | 1.16 | 0.88 | 9.91 | 6.53 | 65.8 | 34.2 | 0.52 | 60.5 | 56.9 | 129.0 |
| 1+50 | 1.77 | 1.13 | 1.15 | 0.84 | 9.73 | 6.08 | 62.6 | 37.4 | 0.60 | 65.0 | 65.7 | 156.0 |
| 1+75 | 1.85 | 1.07 | 1.18 | 0.80 | 10.19 | 5.74 | 56.4 | 43.6 | 0.77 | 68.7 | 73.5 | 184.0 |
| 2+00 | 1.80 | 1.03 | 1.16 | 0.78 | 9.91 | 5.50 | 55.5 | 44.5 | 0.80 | 71.7 | 80.0 | 205.0 |
| 2+25 | 1.80 | 1.00 | 1.16 | 0.76 | 9.91 | 5.34 | 53.8 | 46.2 | 0.86 | 73.9 | 84.9 | 223.0 |
| 2+50 | 1.90 | 0.98 | 1.20 | 0.75 | 10.48 | 5.24 | 50.0 | 50.0 | 1.00 | 75.3 | 88.2 | 235.0 |
| 2+75 | 1.85 | 0.97 | 1.18 | 0.74 | 10.19 | 5.20 | 51.0 | 49.0 | 0.96 | 75.9 | 89.6 | 242.0 |
| 3+00 | 3.52 | 0.97 | 1.59 | 0.74 | 19.80 | 5.19 | 26.2 | 73.8 | 2.82 | 76.0 | 89.8 | 243.0 |
| 3+25 | 4.60 | 0.97 | 1.83 | 0.74 | 26.01 | 5.21 | 20.0 | 80.0 | 4.00 | 75.8 | 89.4 | 242.0 |
| 3+50 | 3.50 | 0.98 | 1.64 | 0.75 | 19.68 | 5.27 | 26.8 | 73.2 | 2.73 | 74.9 | 87.3 | 233.0 |
| 3+75 | 1.97 | 1.02 | 1.23 | 0.78 | 10.88 | 5.46 | 50.2 | 49.8 | 0.99 | 72.3 | 81.3 | 208.0 |
| 4+00 | 1.95 | 1.20 | 1.21 | 0.88 | 10.77 | 6.46 | 60.1 | 39.9 | 0.66 | 61.1 | 58.0 | 132.0 |
| 4+09.5 | 2.17 | 1.26 | 1.30 | 0.91 | 12.03 | 6.82 | 56.6 | 43.4 | 0.77 | 57.9 | 52.1 | 115.0 |

[a] corresponding to the observed velocity.

be no serious objection in allowing the stream to leave the floor of the chute on convex vertical curves, provided the side-walls are of sufficient height to prevent

[13] "Flow of Water in Rivers and Other Channels," by E. Ganguillet and W. R. Kutter, John Wiley & Sons, Inc., 1901, p. 122.

[14] *Engineering News-Record*, January 2, 1936, p. 7.

TABLE 4.—GAGE READINGS AND HYDRAULIC PROPERTIES, RAPID FLUME

| Station | (a) Gage Readings | | | | | | | (b) Hydraulic Properties and Percentage of Air in Water (88 Cu Ft per Sec) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | South Side | | North Side | | Center | | Mean ($y$) | Depths (Ft) | | Hydraulic Radius (Ft) | | Area (Sq Ft) | | Percentages | | Ratio, Col. 15 / Col. 14 | Observed velocity adjusted $V_o$ | Velocity head[b] $H_v$ | $\frac{V^2}{g\,R_c}$ |
| | High | Low | High | Low | High | Low | | Observed $y_o$ | Computed $y_c$ | Observed $R_o$ | Computed $R_c$ | Observed $A_o$ | Computed $A_c$ | Water ($\rho$) | Air ($1-\rho$) | | | | |
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | (15) | (16) | (17) | (18) | (19) |
| 0+50 | 0.68 | 0.58 | 1.91 | 1.71 | 1.12 | 1.02 | 1.28[a] | 1.28 | 0.69 | 0.94 | 0.76 | 8.45 | 4.54 | 53.6 | 46.4 | 0.87 | 19.4 | 5.9 | 15.5 |
| 0+75 | 1.50 | 1.30 | 0.55 | 0.45 | 1.32 | 1.12 | 1.08 | 1.08 | 0.74 | 0.78 | 0.59 | 5.23 | 3.58 | 68.5 | 31.5 | 0.46 | 24.6 | 9.4 | 31.9 |
| 1+00 | 0.99 | 0.89 | 0.75 | 0.55 | 0.83 | 0.78 | 0.80 | 0.80 | 0.64 | 0.59 | 0.50 | 3.67 | 2.94 | 80.2 | 19.8 | 0.25 | 29.9 | 13.9 | 55.6 |
| 1+25 | 1.15 | 1.05 | 0.65 | 0.45 | 0.80 | 0.55 | 0.75 | 0.75 | 0.55 | 0.55 | 0.44 | 3.44 | 2.51 | 73.0 | 27.0 | 0.37 | 35.0 | 19.0 | 86.3 |
| 1+50 | 0.76 | 0.66 | 0.88 | 0.68 | 0.69 | 0.54 | 0.59 | 0.59 | 0.48 | 0.45 | 0.40 | 2.71 | 2.23 | 82.3 | 17.7 | 0.21 | 43.5 | 24.3 | 121.0 |
| 1+75 | 0.79 | 0.64 | 0.66 | 0.46 | 0.72 | 0.57 | 0.64 | 0.64 | 0.44 | 0.50 | 0.37 | 2.94 | 2.02 | 68.8 | 31.2 | 0.45 | 43.5 | 29.4 | 159.0 |
| 2+00 | 0.98 | 0.88 | 0.45 | 0.35 | 0.54 | 0.34 | 0.55 | 0.55 | 0.42 | 0.43 | 0.35 | 2.54 | 1.91 | 75.2 | 24.8 | 0.33 | 46.0 | 32.9 | 188.0 |
| 2+25 | 1.09 | 0.99 | 0.30 | 0.20 | 0.72 | 0.57 | 0.64 | 0.64 | 0.40 | 0.50 | 0.34 | 2.96 | 1.86 | 62.9 | 37.1 | 0.59 | 47.2 | 34.6 | 204.0 |
| 2+50 | 0.86 | 0.76 | 0.87 | 0.77 | 0.62 | 0.52 | 0.69 | 0.69 | 0.40 | 0.54 | 0.34 | 3.18 | 1.82 | 57.2 | 42.8 | 0.75 | 48.4 | 36.4 | 214.0 |
| 2+75 | 0.62 | 0.52 | 0.97 | 0.77 | 0.62 | 0.52 | 0.65 | 0.65 | 0.38 | 0.49 | 0.33 | 2.97 | 1.77 | 59.6 | 40.4 | 0.68 | 49.7 | 38.4 | 221.0 |
| 3+00 | 0.53 | 0.43 | 0.82 | 0.67 | 0.68 | 0.53 | 0.60 | 0.60 | 0.38 | 0.48 | 0.32 | 2.78 | 1.73 | 62.3 | 37.7 | 0.60 | 50.8 | 40.1 | 251.0 |
| 3+25 | 0.89 | 0.79 | 0.67 | 0.62 | 0.62 | 0.52 | 0.66 | 0.66 | 0.37 | 0.50 | 0.32 | 3.02 | 1.72 | 57.0 | 43.0 | 0.75 | 51.2 | 40.8 | 255.0 |
| 3+50 | 0.89 | 0.79 | 0.66 | 0.56 | 0.73 | 0.58 | 0.69 | 0.69 | 0.37 | 0.52 | 0.32 | 3.16 | 1.70 | 53.8 | 46.2 | 0.86 | 51.8 | 41.7 | 261.0 |
| 3+75 | 0.79 | 0.69 | 0.83 | 0.68 | 0.67 | 0.52 | 0.67 | 0.67 | 0.37 | 0.51 | 0.32 | 3.08 | 1.69 | 54.9 | 45.1 | 0.82 | 52.0 | 42.0 | 263.0 |
| 4+00 | 0.80 | 0.70 | 0.77 | 0.67 | 0.83 | 0.63 | 0.74 | 0.74 | 0.37 | 0.56 | 0.32 | 3.40 | 1.70 | 50.0 | 50.0 | 1.00 | 51.8 | 41.7 | 261.0 |
| 4+25 | 0.63 | 0.53 | 0.82 | 0.67 | 0.65 | 0.55 | 0.63 | 0.63 | 0.37 | 0.49 | 0.32 | 2.90 | 1.72 | 59.4 | 40.6 | 0.68 | 51.2 | 40.8 | 255.0 |
| 4+50 | 0.66 | 0.56 | 0.89 | 0.74 | 0.77 | 0.67 | 0.72 | 0.72 | 0.38 | 0.55 | 0.32 | 3.31 | 1.73 | 52.3 | 47.7 | 0.91 | 50.8 | 40.1 | 251.0 |
| 4+75 | 0.72 | 0.62 | 0.75 | 0.65 | 0.73 | 0.63 | 0.68 | 0.68 | 0.39 | 0.52 | 0.33 | 3.13 | 1.78 | 56.9 | 43.1 | 0.76 | 49.4 | 37.9 | 230.0 |
| 5+00 | 0.80 | 0.70 | 0.71 | 0.61 | 0.75 | 0.65 | 0.70 | 0.70 | 0.41 | 0.54 | 0.34 | 3.22 | 1.87 | 58.1 | 41.9 | 0.72 | 47.0 | 34.3 | 202.0 |
| 5+25 | 0.86 | 0.76 | 0.73 | 0.58 | 0.79 | 0.69 | 0.74 | 0.74 | 0.43 | 0.56 | 0.36 | 3.41 | 1.99 | 58.4 | 41.6 | 0.71 | 44.2 | 30.4 | 169.0 |
| 5+50 | 0.79 | 0.64 | 0.92 | 0.77 | 0.83 | 0.73 | 0.78 | 0.78 | 0.46 | 0.58 | 0.39 | 3.59 | 2.13 | 59.3 | 40.7 | 0.68 | 41.3 | 26.5 | 136.0 |
| 5+75 | 0.75 | 0.60 | 0.96 | 0.81 | 0.83 | 0.73 | 0.78 | 0.78 | 0.49 | 0.58 | 0.40 | 3.59 | 2.25 | 62.8 | 37.2 | 0.59 | 39.0 | 23.6 | 118.0 |
| 6+00 | 1.04 | 0.89 | 0.77 | 0.67 | 0.89 | 0.79 | 0.84 | 0.84 | 0.52 | 0.62 | 0.42 | 3.86 | 2.38 | 61.7 | 38.3 | 0.62 | 37.0 | 21.3 | 101.0 |
| 6+25 | 0.84 | 0.69 | 0.80 | 0.70 | 0.81 | 0.71 | 0.76 | 0.76 | 0.54 | 0.57 | 0.44 | 3.50 | 2.48 | 70.9 | 29.1 | 0.41 | 35.0 | 19.6 | 89.1 |
| 6+50 | 0.78 | 0.68 | 0.85 | 0.70 | 0.80 | 0.70 | 0.75 | 0.75 | 0.56 | 0.57 | 0.45 | 3.45 | 2.56 | 72.1 | 27.9 | 0.39 | 34.4 | 18.4 | 81.7 |
| 6+67 | 0.93 | 0.78 | 0.92 | 0.77 | 0.89 | 0.79 | 0.84 | 0.84 | 0.57 | 0.61 | 0.46 | 3.86 | 2.64 | 68.5 | 31.5 | 0.46 | 33.3 | 17.2 | 74.8 |

[a] Additional reading taken on this section to determine mean. [b] Velocity head, $H_v$, corresponding to the observed velocity.

overtopping by the spray. The observations at the Hat Creek chute indicate that the spreading of the jet over a greater cross section increased the friction and aided in the dissipation of the energy. This fact may be useful in design, where it is desirable to reduce the velocity of the stream before discharging the jet from the end of the chute.

Since water traveling in the chutes is always at less than the critical depth, changes in horizontal direction are more difficult to make and, if possible, should be avoided. If a horizontal curve becomes necessary, it should preferably be inserted near the head of the chute before the water has gathered much velocity. Model tests should be made to test the design prior to actual construction.

TABLE 5.—VELOCITY OBSERVATIONS, RAPID FLUME

| Full Run Stations | | Time (Sec) | | Partial Run Stations | | Time (Sec) | | Ratio, Col. 7 / Col. 8 | Partial Run Stations | | Time (Sec) | | Ratio, Col. 7 / Col. 8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Upper | Lower | Observed[a] | Corrected | Upper | Lower | Observed | Equivalent full run | | Upper | Lower | Observed | Equivalent full run | |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (5) | (6) | (7) | (8) | (9) |
| 0+50 | 1+25 | 2.88 | 2.88 | 2+25 | 4+25 | 3.93 | 4.18 | 94.0 | 2+25 | 3+25 | 2.00 | 2.09 | 95.5 |
| 0+50 | 2+25 | 5.26 | 5.23 | 2+25 | 5+25 | 6.06 | 6.61 | 91.5 | 2+25 | 4+25 | 3.97 | 4.18 | 95.0 |
| 0+50 | 3+25 | 7.35 | 7.22 | 2+25 | 6+67 | 10.40 | 10.85 | 96.0 | 3+25 | 4+25 | 1.94 | 2.09 | 93.0 |
| 0+50 | 4+25 | 9.44 | 9.15 | 2+25 | ..... | .... | .... | .... | 3+25 | 5+25 | 3.98 | 4.52 | 88.0 |
| 0+50 | 5+25 | 11.87 | 11.15 | 3+25 | 4+25 | 1.84 | 2.09 | 88.0 | 4+25 | 5+25 | 2.00 | 2.43 | 82.0 |
| 0+50 | 6+67 | 16.11 | 14.94 | 3+25 | 5+25 | 4.08 | 4.52 | 90.0 | 4+25 | 6+67 | 5.90 | 6.67 | 88.5 |
| ..... | ..... | .... | .... | 3+25 | 6+67 | 8.20 | 8.76 | 93.5 | 5+25 | 6+67 | 3.72 | 4.24 | 88.0 |

[a] Observed times have been corrected by factors determined for length of travel.

The water surface across the channel of a chute constructed on a curve will be tangent to a parabola.[15] Of course, this is assuming that the channel is relatively narrow permitting the use of the radius of the center line $r$ as the average curvature of the section. Then, the inclination of this tangent with the horizontal is

$$\tan \psi = \frac{V^2}{g\,r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

Verification of this formula was possible on the South Canal chute, where the difference in elevation of the water surface between the inner and outer walls approximated the slope $\psi$. In model tests[16] conducted by the East Bay Municipal Utility District, Robert C. Kennedy, M. Am. Soc. C. E., has demonstrated that the flow of water around a horizontal curve is greatly improved if the bottom of the flume is superelevated on the outer wall in accordance with this formula.

## CALCULATION OF RESULTS OF TESTS

The results of the depth measurements and velocity observations are given in Tables 1 to 11. The gage readings are recorded for both the highest and

[15] *Text Book of Hydraulics*, by G. E. Russell, Henry Holt & Co., 4th Ed., p. 49.

[16] *Western Construction News*, April, 1942, p. 158.

TABLE 6.—GAGE READINGS, SOUTH CANAL CHUTE

| Station | Left Side | | Right Side | | Mean ($y$) |
|---|---|---|---|---|---|
| | High (1) | Low (2) | High (3) | Low (4) | (5) |
| **363.5 Cu Ft per Sec** | | | | | |
| 0+00 | 2.0 | 1.9 | 2.1 | 1.9 | 1.98 |
| 0+50 | ... | ... | 1.8 | 1.6 | 1.74 |
| 1+00 | 1.7 | 1.5 | 1.8 | 1.5 | 1.63 |
| 1+50 | 1.6 | 1.4 | 1.8 | 1.5 | 1.59 |
| 2+00 | 1.6 | 1.4 | 1.7 | 1.3 | 1.50 |
| 2+50 | 1.5 | 1.3 | 1.8 | 1.4 | 1.50 |
| 3+00 | 1.1 | 0.9 | 2.3 | 1.9 | 1.55 |
| 3+50 | 1.2 | 1.0 | 2.1 | 1.5 | 1.45 |
| 4+00 | 1.2 | 0.8 | 2.3 | 1.9 | 1.50 |
| 4+50 | 1.7 | 1.4 | 1.5 | 1.1 | 1.43 |
| 5+00 | 1.4 | 1.2 | 1.7 | 1.3 | 1.40 |
| 5+50 | 1.6 | 1.3 | 1.7 | 1.3 | 1.47 |
| 6+00 | 1.8 | 1.5 | 1.7 | 1.3 | 1.58 |
| 6+50 | 1.9 | 1.5 | 1.6 | 1.1 | 1.52 |
| 7+00 | 1.9 | 1.6 | 1.5 | 0.9 | 1.48 |
| 7+50 | 1.85 | 1.5 | 1.8 | 1.4 | 1.64 |
| 8+00 | 1.8 | 1.4 | 2.3 | 1.7 | 1.80 |
| 8+50 | 1.7 | 1.4 | 1.6 | 1.1 | 1.45 |
| 9+00 | 2.0 | 1.7 | 1.7 | 1.3 | 1.68 |
| 9+50 | 1.8 | 1.5 | 1.9 | 1.4 | 1.65 |
| 10+00 | 1.7 | 1.4 | 2.0 | 1.5 | 1.65 |
| 10+50 | 1.8 | 1.5 | 2.1 | 1.6 | 1.70 |
| 11+00 | 1.9 | 1.7 | 1.8 | 1.4 | 1.70 |
| 11+50 | 1.9 | 1.7 | 1.8 | 1.4 | 1.70 |
| 12+00 | 1.9 | 1.5 | 2.0 | 1.7 | 1.78 |
| 12+50 | 1.9 | 1.6 | 1.9 | 1.5 | 1.72 |
| 13+00 | 1.9 | 1.6 | 1.8 | 1.4 | 1.68 |
| 13+50 | 2.0 | 1.7 | 1.8 | 1.4 | 1.72 |
| 14+00 | 1.9 | 1.5 | 1.9 | 1.5 | 1.70 |
| 14+50 | 2.0 | 1.6 | 2.1 | 1.7 | 1.85 |
| 15+00 | 2.2 | 1.8 | 1.8 | 1.4 | 1.80 |
| 15+50 | 2.2 | 1.8 | 1.9 | 1.4 | 1.82 |
| 15+75 | 1.7 | 1.4 | 1.6 | 1.3 | 1.50 |
| 16+00 | 2.2 | 1.9 | 2.0 | 1.7 | 1.95 |
| **334 Cu Ft per Sec** | | | | | |
| 0+00 | ... | ... | 1.9 | 1.7 | 1.78 |
| 0+50 | ... | ... | 1.6 | 1.5 | 1.70 |
| 1+00 | 1.6 | 1.4 | 1.5 | 1.3 | 1.45 |
| 1+50 | 1.6 | 1.3 | 1.6 | 1.3 | 1.45 |
| 2+00 | 1.7 | 1.4 | 1.5 | 1.2 | 1.45 |
| 2+50 | 1.6 | 1.2 | 1.6 | 1.3 | 1.42 |
| 3+00 | 1.1 | 0.8 | 2.1 | 1.8 | 1.45 |
| 3+50 | 1.4 | 1.0 | 1.8 | 1.4 | 1.40 |
| 4+00 | 1.2 | 0.8 | 2.1 | 1.7 | 1.45 |
| 4+50 | 1.7 | 1.4 | 1.4 | 1.1 | 1.40 |
| 5+00 | 1.4 | 1.1 | 1.5 | 1.2 | 1.30 |
| 5+50 | 1.6 | 1.3 | 1.5 | 1.2 | 1.40 |
| 6+00 | 1.8 | 1.4 | 1.5 | 1.2 | 1.48 |
| 6+50 | 1.9 | 1.5 | 1.5 | 1.1 | 1.50 |
| 7+00 | 2.2 | 1.7 | 1.3 | 0.9 | 1.52 |
| 7+50 | 2.0 | 1.5 | 1.7 | 1.3 | 1.62 |
| 8+00 | 1.9 | 1.4 | 2.1 | 1.6 | 1.75 |
| 8+50 | 1.7 | 1.3 | 1.6 | 1.1 | 1.42 |
| 9+00 | 1.9 | 1.6 | 1.7 | 1.3 | 1.62 |
| 9+50 | 1.7 | 1.4 | 1.7 | 1.3 | 1.52 |
| 10+00 | 1.7 | 1.4 | 1.9 | 1.5 | 1.62 |
| 10+50 | 1.7 | 1.4 | 1.9 | 1.6 | 1.65 |

| Station | Left Side | | Right Side | | Mean ($y$) |
|---|---|---|---|---|---|
| | High (1) | Low (2) | High (3) | Low (4) | (5) |
| **334 Cu Ft per Sec** | | | | | |
| 11+00 | 1.9 | 1.6 | 1.7 | 1.3 | 1.62 |
| 11+50 | 1.9 | 1.6 | 1.65 | 1.3 | 1.62 |
| 12+00 | 1.9 | 1.6 | 1.8 | 1.5 | 1.70 |
| 12+50 | 1.8 | 1.4 | 1.8 | 1.4 | 1.60 |
| 13+00 | 1.8 | 1.5 | 1.7 | 1.4 | 1.60 |
| 13+50 | 1.8 | 1.5 | 1.7 | 1.3 | 1.58 |
| 14+00 | 1.7 | 1.3 | 1.7 | 1.4 | 1.52 |
| 14+50 | 1.7 | 1.4 | 1.8 | 1.4 | 1.58 |
| 15+00 | 1.9 | 1.6 | 1.6 | 1.3 | 1.60 |
| 15+50 | 1.9 | 1.6 | 1.7 | 1.4 | 1.65 |
| 15+75 | 1.4 | 1.0 | 1.4 | 1.1 | 1.22 |
| 16+00 | 1.9 | 1.6 | 1.8 | 1.4 | 1.68 |
| 16+50 | 2.0 | 1.7 | 2.0 | 1.7 | 1.85 |
| 17+00 | 2.1 | 1.8 | 2.0 | 1.7 | 1.90 |
| 17+50 | 2.2 | 1.9 | 2.2 | 1.9 | 2.05 |
| 18+00 | 2.4 | 2.1 | 2.2 | 1.9 | 2.15 |
| 18+50 | 2.6 | 2.3 | 2.1 | 1.9 | 2.22 |
| 19+00 | 2.3 | 2.1 | 2.3 | 2.1 | 2.20 |
| 19+50 | 2.4 | 2.2 | 2.3 | 2.1 | 2.25 |
| 20+00 | 2.3 | 2.0 | 2.6 | 2.3 | 2.30 |
| 20+50 | 2.6 | 2.3 | 2.3 | 2.1 | 2.32 |
| 21+00 | 2.2 | 2.0 | 2.7 | 2.5 | 2.35 |
| 21+50 | 2.9 | 2.7 | 1.9 | 1.7 | 2.30 |
| 22+00 | 2.7 | 2.4 | 2.4 | 2.2 | 2.42 |
| 22+50 | 2.7 | 2.4 | 2.1 | 1.9 | 2.27 |
| 23+00 | 2.0 | 1.75 | 2.9 | 2.6 | 2.31 |
| 23+25 | 2.3 | 2.0 | 2.5 | 2.3 | 2.28 |
| 23+50 | 1.7 | 1.5 | 3.0 | 2.8 | 2.25 |
| 23+75 | 1.8 | 1.6 | 2.7 | 2.5 | 2.15 |
| 24+00 | 1.5 | 1.3 | 2.8 | 2.4 | 2.00 |
| 24+25 | 1.5 | 1.3 | 2.7 | 2.4 | 1.98 |
| 24+50 | 1.3 | 1.1 | 2.9 | 2.6 | 1.98 |
| 24+75 | 1.3 | 1.1 | 2.6 | 2.4 | 1.85 |
| 25+00 | 0.8 | 0.6 | 3.7 | 3.2 | 2.08 |
| 25+25 | 0.8 | 0.6 | 3.5 | 3.2 | 2.03 |
| 25+50 | 1.0 | 0.8 | 3.1 | 2.7 | 1.95 |
| 26+00 | 1.7 | 1.4 | 2.6 | 2.2 | 1.98 |
| 26+50 | 2.2 | 1.9 | 2.4 | 2.2 | 2.18 |
| 27+00 | 2.8 | 2.3 | 1.7 | 1.2 | 2.00 |
| 27+50 | 2.4 | 2.1 | 2.1 | 1.7 | 2.08 |
| 28+00 | 2.4 | 2.0 | 2.2 | 1.9 | 2.13 |
| 28+50 | 2.5 | 2.2 | 2.3 | 2.0 | 2.25 |
| 29+00 | 2.1 | 1.8 | 2.5 | 2.2 | 2.15 |
| 29+50 | 2.1 | 1.8 | 2.6 | 2.3 | 2.20 |
| 30+00 | 2.1 | 1.8 | 2.7 | 2.3 | 2.22 |
| 30+50 | 2.4 | 2.1 | 2.3 | 2.0 | 2.20 |
| 31+00 | 2.6 | 2.3 | 2.3 | 2.0 | 2.30 |
| 31+50 | 2.5 | 2.2 | 2.3 | 2.0 | 2.25 |
| 32+00 | 2.7 | 2.4 | 2.1 | 1.8 | 2.25 |
| 32+50 | 2.5 | 2.2 | 2.0 | 1.7 | 2.10 |
| 32+60 | 2.4 | 2.2 | 2.0 | 1.7 | 2.08 |
| 33+20 | 2.6 | 2.3 | 1.9 | 1.6 | 2.10 |
| 33+50 | 2.2 | 1.9 | 2.4 | 2.1 | 2.15 |
| 34+00 | 2.0 | 1.8 | 2.7 | 2.3 | 2.20 |
| 34+50 | 2.4 | 2.2 | 2.1 | 1.8 | 2.12 |
| 35+00 | 2.6 | 2.3 | 1.8 | 1.5 | 2.08 |
| 35+20 | 2.9 | 2.5 | 1.5 | 1.2 | 2.03 |
| 35+35 | 2.0 | 1.6 | 2.3 | 2.0 | 1.97 |

lowest point observed at each station, together with the mean depth calculated from these observations. In the case of Rapid Flume, readings are given for depth in the center of the flume for the upper part of the channel where cross waves developed due to the curvature of the flume. The results for Hat Creek chute are given in Table 1, for Rapid Flume in Table 4(*a*), and for the South Canal chute in Table 6.

In making velocity tests, it was better to time the color over a rather short stretch of flume, because, if the measuring course were too long, the color cloud spread out over such a long distance that it was difficult to estimate the time of passing of the center of the color band. On the first series of observations color was added at the head of the chute and the time of travel was observed over courses of increasing length by having the timers moved progressively to stations farther down the chute. An analysis of these results indicated that the observed time at the lower stations was too great. On the latter tests the color was dropped in at various places along the chute and the velocity was timed over courses not exceeding 100 ft in length where conditions would permit. On the South Canal chute the great length of channel and slower velocities indicated that an increased length of course could be used without error. The results obtained from separate courses on these tests were totaled so as to give the entire time for the water to travel from the inlet to the outlet of the chute. In all cases the sum of the times taken over the short courses was less than the observed time over the total course. This anomalous condition can be explained readily.

TABLE 7.—VELOCITY OBSERVATIONS, FULL RUN STATIONS, SOUTH CANAL CHUTE

| STATIONS | | TIME (SEC) | |
|---|---|---|---|
| Upper (1) | Lower (2) | Observed[a] (3) | Corrected (4) |
| 1+00 | 2+65 | 5.12 | 5.12 |
| 1+00 | 3+95 | 9.41 | 9.24 |
| 1+00 | 6+20 | 16.41 | 15.58 |
| 1+00 | 8+00 | 22.93 | 20.70 |
| 8+00 | 9+94 | 7.02 | 7.02 |
| 8+00 | 10+65 | 9.75 | 9.61 |
| 8+00 | 14+00 | 23.30 | 22.01 |
| 10+65 | 17+00 | 28.56 | 26.82 |
| 17+00 | 19+50 | 13.22 | 13.22 |
| 17+00 | 21+50 | 26.02 | 24.85 |
| 17+00 | 23+00 | 36.36 | 33.39 |
| 23+00 | 25+00 | 7.65 | 7.65 |
| 25+00 | 27+00 | 5.93 | 5.93 |
| 25+00 | 28+00 | 9.40 | 9.21 |
| 28+00 | 30+00 | 7.58 | 7.58 |
| 28+00 | 32+00 | 15.93 | 15.46 |
| 30+00 | 32+00 | 7.84 | 7.84 |
| 30+00 | 33+50 | 14.07 | 13.72 |
| 32+00 | 35+35 | 12.47 | 12.22 |

[a] Observed times have been corrected by factors determined for length of travel.

The color cloud was retarded along the side-walls as it proceeded down the chute and became attenuated with increasing distances of travel. In observing the velocities, the time was taken to the approximate center of the color cloud as closely as it could be judged by eye. Along a 400-ft course the color cloud extended between 75 and 100 ft. Analyses of the results indicate that probably the greatest concentration of color was at the head of the slug; hence, the time lapse recorded to the center of the cloud was too long. On shorter courses the color cloud stayed together better, and less error, therefore, was introduced by estimating the location of the center by eye. The results of these observations showing the variation in time over long and short courses have been tabulated

for Hat Creek chute in Table 2, Rapid Flume in Table 5, and for the South Canal chute in Tables 7 and 7*A*. These observations on short courses were used to obtain factors applied to correct inaccuracies in the observed time over the longer courses. Since for calculation purposes the depth readings needed to be recorded for the perpendicular distance, corrections were made by multiplying the observed depth taken perpendicular to the inclination of the chute by the cosine of the angle of the slope.

TABLE 7*A*.—VELOCITY OBSERVATIONS, PARTIAL RUN STATIONS, SOUTH CANAL SHUTE

| STATIONS | | TIME (SEC) | | Ratio, Col. 3 / Col. 4 |
|---|---|---|---|---|
| Upper | Lower | Observed | Equivalent full run | |
| (1) | (2) | (3) | (4) | (5) |
| 3+95 | 6+20 | 6.79 | 7.00 | 97.0 |
| 3+95 | 8+00 | 13.30 | 13.52 | 98.5 |
| 10+65 | 14+00 | 13.29 | 13.55 | 98.5 |
| 21+50 | 23+00 | 8.57 | 10.34 | 83.0 |
| 30+00 | 32+00 | 7.84 | 8.35 | 94.0 |
| 32+00 | 33+50 | 5.28 | 6.23 | 85.0 |

The rapid acceleration of the velocity, particularly near the head of the chutes, required a segregation of the time over shorter courses than was practical for timing. Therefore, an estimated subdivision of the time from the observations for purposes of subsequent computation was made for 25-ft intervals on Hat Creek chute and Rapid Flume and for 50-ft intervals on the South Canal chute. In making this subdivision, the times on the various tests were compared graphically and increments for the 25-ft or 50-ft intervals estimated from the elapsed time over the longer courses.

A comparison of the recorded observations of depth and velocity illustrates forcibly the difficulty in obtaining precise results in measuring flows at high velocities where air is entrained with the water.

The velocity head for the adjusted mean velocity in each 25-ft or 50-ft section was plotted on the profile of the respective chutes (see Figs. 1 and 6 for Hat Creek chute and Rapid Flume), and the energy gradient was drawn. From this gradient line the velocity head at each station was scaled and the corresponding velocity calculated. These velocities were used in conjunction with the observed areas to compute the coefficient of retardation. In addition, the observed velocity and energy gradient were used in conjunction with the area computed by the relation $A_c = \frac{Q}{V_o}$ and the corresponding hydraulic radii to calculate retardation factors. The velocity at each station was also computed by the equation $V_c = \frac{Q}{A_o}$, using for the area the values computed from the observed depths, including water plus entrained air. These velocity heads, which were smaller than the observed values, were plotted on graphs and the corresponding energy slopes determined. Necessarily these slopes were greater in the zone of accelerated flow than those obtained by use of observed velocity

heads. These three methods of computation were used by Fred C. Scobey, M. Am. Soc. C. E., in his study of flow of water in chutes.[17]

In his bulletin the computation of the retardation factor has been made by taking the energy gradient parallel to the bottom slope. This may be correct in many instances, but in the tests described in this paper it was found that the energy slope was seldom parallel to the bottom slope, as can be seen by an examination of Figs. 1 and 6.

These elements of the water prism obtained in the manner just described were used for the determination of the coefficients of the flow in the Chézy formula and of the retardation factors in Kutter's and Manning's formulas by each of the three methods described; namely, by using (1) the observed velocities and the observed area, (2) the observed velocity and areas computed as $\frac{Q}{V_o}$, and (3) the observed areas and velocities computed as $\frac{Q}{A_o}$.

The hydraulic properties of the water prism in the three chutes tested together with the percentage of air in the water and the value of the parameter $\frac{V^2}{g\,R_c}$ were tabulated for each flow observed. These results are given in Table 3 for Hat Creek chute, in Table 4(*b*) for Rapid Flume, and in Table 8 for South Canal chute.

*Bureau of Reclamation Tests on Kittitas Chute.*—In 1938 the Bureau of Reclamation made tests of air entrainment on the wasteway from the Kittitas Main Canal located on the Yakima Project in a western state. This chute discharges water from the main canal into the Yakima River. The flow of water from the canal into the inlet of the chute is controlled by a radial gate. The total length of the chute is 1,301 ft measured on the slope, and the total drop from the canal to the bottom of the bucket at the Yakima River is 340 ft. The first 753 ft of the chute are on a 10° 12′ slope and the initial 318 ft are built on a taper section. At the end of this slope there is a convex vertical curve about 90 ft in length connecting with a steep slope of 33° 10′, having a length of 289 ft. This joins with a concave vertical curve about 90 ft in length, joining an almost horizontal tangent with a slight downward slope of 0° 04′ for a distance of 79 ft to the end of the chute. The jet at the end of the incline discharges into the Yakima River above the maximum high-water level.

The concrete of the chute is smooth on the side-walls, but judging from photographs, the bottom appears to be roughened slightly, probably due to sand carried down the chute with the discharge. The width of the channel decreases from 20 ft at the inlet to 8 ft at the end of the taper section. From this point a uniform width of 8 ft is carried for the remaining length of the chute. The side-walls are 7 ft high, except at the inlet and near the outlet.

Discharge measurements were taken with a current meter in the canal above the chute. Measurements of the depth to the water surface in the incline were taken with a point gage, reading the distance down from the top of the side-walls. A traverse of the channel was made, taking observations at the side-walls and at every foot across the section. The first point gage station

[17] "The Flow of Water in Flumes," by Fred C. Scobey, *Technical Bulletin No. 393*, U. S. Dept. of Agriculture, pp. 89–91.

was located at a distance of 343 ft from the inlet, a short distance below the end of the taper section. Depth readings were taken at approximately 100-ft intervals for a distance of 410 ft along the 10° 12′ slope, for 210 ft along the 33° 10′ slope, and for 50 ft along the 0° 04′ slope. The depth and area at each of the ten observation stations were calculated from these measurements.

Velocity observations were taken electrically, using salt as a conductor, balls of moistened flour salt being dropped into the chute for each test run. The velocity was recorded on an oscillograph through electrodes installed in the walls of the chute. The oscillograph consisted essentially of three moving coil galvanometers, an optical system, and a motor-driven camera. Two of the galvanometers recorded the passage of the salt cloud past the upper and lower stations. The third was connected to a 60-cycle alternating current and recorded a time scale on the film.

This method has the advantage of affording a permanent record of the test. Nevertheless, it was difficult to determine the center of the gravity of the salt cloud as it passed each station. Also, it was noted on the film that the flow, particularly on the steep grade, occurred in surges, with the result

TABLE 8.—HYDRAULIC PROPERTIES, SOUTH CANAL CHUTE

| Station | Hydraulic Properties and Percentage of Air in Water | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Depths (Ft) | | Hydraulic Radius (Ft) | | Area (Sq Ft) | | Percentages | | Ratio, Col. 8 / Col. 7 | Observed velocity $V_o$ | Velocity head[a] $H_v$ | $\frac{V^2}{g R_c}$ |
| | Observed $y_o$ | Computed $y_c$ | Observed $R_o$ | Computed $R_c$ | Observed $A_o$ | Computed $A_c$ | Water ($\rho$) | Air ($1-\rho$) | | | | |
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) |
| (a) Discharge 363.5 Cu Ft per Sec | | | | | | | | | | | | |
| 0+00 | 1.98 | 2.12 | 1.46 | 1.55 | 19.36 | 20.7 | 106.9 | -06.9 | .... | 17.6 | 4.8 | 6.2 |
| 0+50 | 1.74 | 1.33 | 1.34 | 1.08 | 16.96 | 12.9 | 76.1 | 23.9 | 0.31 | 28.2 | 12.4 | 23.0 |
| 1+00 | 1.63 | 1.24 | 1.27 | 1.02 | 15.86 | 11.95 | 75.3 | 24.7 | 0.33 | 30.4 | 14.4 | 28.2 |
| 1+50 | 1.59 | 1.17 | 1.25 | 0.98 | 15.46 | 11.3 | 73.1 | 26.3 | 0.36 | 32.1 | 16.0 | 32.7 |
| 2+00 | 1.50 | 1.16 | 1.19 | 0.97 | 14.56 | 11.15 | 76.6 | 23.4 | 0.31 | 32.6 | 16.5 | 34.0 |
| 2+50 | 1.50 | 1.16 | 1.19 | 0.97 | 14.56 | 11.15 | 76.6 | 23.4 | 0.31 | 32.6 | 16.5 | 34.0 |
| 3+00 | 1.55 | 1.16 | 1.22 | 0.97 | 15.06 | 11.15 | 74.0 | 26.0 | 0.35 | 32.6 | 16.5 | 34.0 |
| 3+50 | 1.45 | 1.16 | 1.16 | 0.97 | 14.06 | 11.15 | 79.3 | 20.7 | 0.26 | 32.6 | 16.5 | 34.0 |
| 4+00 | 1.50 | 1.16 | 1.19 | 0.97 | 14.56 | 11.15 | 76.6 | 23.4 | 0.31 | 32.6 | 16.5 | 34.0 |
| 4+50 | 1.43 | 1.16 | 1.15 | 0.97 | 13.86 | 11.15 | 80.4 | 19.6 | 0.24 | 32.6 | 16.5 | 34.0 |
| 5+00 | 1.40 | 1.16 | 1.13 | 0.97 | 13.56 | 11.15 | 82.2 | 17.8 | 0.22 | 32.6 | 16.5 | 34.0 |
| 5+50 | 1.47 | 1.16 | 1.17 | 0.97 | 14.26 | 11.15 | 78.2 | 21.8 | 0.28 | 32.6 | 16.5 | 34.0 |
| 6+00 | 1.58 | 1.17 | 1.24 | 0.98 | 15.36 | 11.25 | 73.2 | 26.8 | 0.36 | 31.4 | 15.3 | 31.2 |
| 6+50 | 1.52 | 1.20 | 1.20 | 1.00 | 14.76 | 11.6 | 78.6 | 21.4 | 0.27 | 31.3 | 15.2 | 30.4 |
| 7+00 | 1.48 | 1.24 | 1.18 | 1.03 | 14.36 | 12.0 | 83.6 | 16.4 | 0.20 | 30.3 | 14.3 | 27.8 |
| 7+50 | 1.64 | 1.29 | 1.28 | 1.06 | 15.96 | 12.5 | 78.3 | 21.7 | 0.28 | 29.1 | 13.2 | 24.9 |
| 8+00 | 1.80 | 1.36 | 1.37 | 1.10 | 17.56 | 13.15 | 74.9 | 25.1 | 0.34 | 27.6 | 11.8 | 21.5 |
| 8+50 | 1.45 | 1.36 | 1.16 | 1.10 | 14.06 | 13.15 | 93.5 | 6.5 | 0.07 | 27.6 | 11.8 | 21.5 |
| 9+00 | 1.68 | 1.36 | 1.30 | 1.10 | 16.36 | 13.15 | 80.4 | 19.6 | 0.24 | 27.6 | 11.8 | 21.5 |
| 9+50 | 1.65 | 1.36 | 1.28 | 1.10 | 16.06 | 13.15 | 81.9 | 18.1 | 0.22 | 27.6 | 11.8 | 21.5 |
| 10+00 | 1.65 | 1.36 | 1.28 | 1.10 | 16.06 | 13.15 | 81.9 | 18.1 | 0.22 | 27.6 | 11.8 | 21.5 |
| 10+50 | 1.70 | 1.37 | 1.31 | 1.11 | 16.56 | 13.3 | 80.3 | 19.7 | 0.25 | 27.3 | 11.6 | 20.9 |
| 11+00 | 1.70 | 1.38 | 1.31 | 1.12 | 16.56 | 13.4 | 80.9 | 19.1 | 0.24 | 27.1 | 11.4 | 20.4 |
| 11+50 | 1.70 | 1.38 | 1.31 | 1.12 | 16.56 | 13.4 | 80.9 | 19.1 | 0.24 | 27.1 | 11.4 | 20.4 |
| 12+00 | 1.78 | 1.40 | 1.36 | 1.13 | 17.36 | 13.55 | 78.1 | 21.9 | 0.28 | 26.8 | 11.2 | 19.8 |
| 12+50 | 1.72 | 1.40 | 1.32 | 1.13 | 16.76 | 13.6 | 81.1 | 18.9 | 0.23 | 26.7 | 11.1 | 19.6 |
| 13+00 | 1.68 | 1.40 | 1.30 | 1.13 | 16.36 | 13.6 | 83.1 | 16.9 | 0.20 | 26.7 | 11.1 | 19.6 |
| 13+50 | 1.72 | 1.41 | 1.32 | 1.14 | 16.76 | 13.65 | 81.4 | 18.6 | 0.23 | 26.6 | 10.8 | 18.9 |
| 14+00 | 1.70 | 1.42 | 1.31 | 1.14 | 16.56 | 13.75 | 83.0 | 17.0 | 0.20 | 26.4 | 10.8 | 18.9 |

TABLE 8—(*Continued*)

| Station | Hydraulic Properties and Percentage of Air in Water | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Depths (Ft) | | Hydraulic Radius (Ft) | | Area (Sq Ft) | | Percentages | | Ratio, Col. 8 / Col. 7 | Observed velocity $V_o$ | Velocity head[a] $H_o$ | $\frac{V^2}{g R_o}$ |
| | Observed $y_o$ | Computed $y_c$ | Observed $R_o$ | Computed $R_c$ | Observed $A_o$ | Computed $A_c$ | Water ($\rho$) | Air ($1-\rho$) | | | | |
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) |
| (*b*) Discharge 334 Cu Ft per Sec | | | | | | | | | | | | |
| 10+50 | 1.65 | 1.36 | 1.28 | 1.10 | 16.06 | 13.15 | 81.9 | 18.1 | 0.22 | 25.4 | 10.0 | 18.1 |
| 11+00 | 1.62 | 1.36 | 1.26 | 1.10 | 15.76 | 13.15 | 83.4 | 16.6 | 0.20 | 25.4 | 10.0 | 18.1 |
| 11+50 | 1.62 | 1.37 | 1.27 | 1.11 | 15.76 | 13.25 | 84.1 | 15.9 | 0.19 | 25.2 | 9.9 | 17.8 |
| 12+00 | 1.70 | 1.38 | 1.31 | 1.12 | 16.56 | 13.35 | 80.0 | 20.0 | 0.25 | 25.0 | 9.7 | 17.3 |
| 12+50 | 1.60 | 1.38 | 1.25 | 1.12 | 15.56 | 13.35 | 85.8 | 14.2 | 0.17 | 25.0 | 9.7 | 17.3 |
| 13+00 | 1.60 | 1.39 | 1.25 | 1.12 | 15.56 | 13.45 | 86.4 | 13.6 | 0.16 | 24.8 | 9.6 | 17.1 |
| 13+50 | 1.57 | 1.39 | 1.23 | 1.12 | 15.26 | 13.5 | 88.5 | 11.5 | 0.13 | 24.7 | 9.5 | 17.0 |
| 14+00 | 1.53 | 1.40 | 1.21 | 1.13 | 14.86 | 13.55 | 91.2 | 8.8 | 0.10 | 24.6 | 9.4 | 16.6 |
| 14+50 | 1.58 | 1.48 | 1.24 | 1.18 | 15.36 | 14.4 | 93.8 | 6.2 | 0.06 | 23.2 | 8.4 | 14.2 |
| 15+00 | 1.60 | 1.51 | 1.25 | 1.20 | 15.56 | 14.7 | 94.5 | 5.5 | 0.06 | 22.7 | 8.0 | 13.3 |
| 15+50 | 1.65 | 1.53 | 1.28 | 1.21 | 16.06 | 14.9 | 92.8 | 7.2 | 0.08 | 22.4 | 7.8 | 12.9 |
| 16+00 | 1.68 | 1.57 | 1.30 | 1.24 | 16.36 | 15.3 | 93.5 | 6.5 | 0.07 | 21.8 | 7.4 | 11.9 |
| 16+50 | 1.85 | 1.64 | 1.40 | 1.28 | 18.06 | 16.0 | 88.6 | 11.4 | 0.13 | 20.9 | 6.8 | 10.6 |
| 17+00 | 1.90 | 1.80 | 1.43 | 1.37 | 18.56 | 17.6 | 94.8 | 5.2 | 0.05 | 19.0 | 5.6 | 8.2 |
| 17+50 | 2.05 | 1.80 | 1.51 | 1.37 | 20.06 | 17.6 | 87.7 | 12.3 | 0.14 | 19.0 | 5.6 | 8.2 |
| 18+00 | 2.15 | 1.80 | 1.56 | 1.37 | 21.06 | 17.6 | 83.6 | 16.4 | 0.20 | 19.0 | 5.6 | 8.2 |
| 18+50 | 2.22 | 1.80 | 1.60 | 1.37 | 21.76 | 17.6 | 80.9 | 19.1 | 0.24 | 19.0 | 5.6 | 8.2 |
| 19+00 | 2.20 | 1.80 | 1.59 | 1.37 | 21.56 | 17.6 | 81.6 | 18.4 | 0.23 | 19.0 | 5.6 | 8.2 |
| 19+50 | 2.25 | 1.80 | 1.61 | 1.37 | 22.06 | 17.6 | 79.8 | 20.2 | 0.25 | 19.0 | 5.6 | 8.2 |
| 20+00 | 2.30 | 1.91 | 1.64 | 1.43 | 22.56 | 18.65 | 82.7 | 17.3 | 0.21 | 17.9 | 5.0 | 7.0 |
| 20+50 | 2.32 | 1.91 | 1.65 | 1.43 | 22.76 | 18.65 | 81.9 | 18.1 | 0.22 | 17.9 | 5.0 | 7.0 |
| 21+00 | 2.35 | 1.98 | 1.66 | 1.46 | 23.06 | 19.4 | 84.1 | 15.9 | 0.19 | 17.2 | 4.6 | 6.3 |
| 21+50 | 2.30 | 2.01 | 1.64 | 1.49 | 22.56 | 19.65 | 87.1 | 12.9 | 0.15 | 17.0 | 4.5 | 6.0 |
| 22+00 | 2.42 | 2.01 | 1.69 | 1.49 | 23.76 | 19.65 | 82.7 | 16.3 | 0.20 | 17.0 | 4.5 | 6.0 |
| 22+50 | 2.27 | 2.05 | 1.62 | 1.51 | 22.26 | 20.10 | 91.1 | 8.9 | 0.10 | 16.6 | 4.3 | 5.7 |
| 23+00 | 2.31 | 2.05 | 1.64 | 1.51 | 20.06 | 20.10 | 92.8 | 7.2 | 0.08 | 16.6 | 4.3 | 5.7 |
| 23+50[b] | 2.25 | 2.26 | 1.43 | 1.43 | 15.31 | 15.40 | 100.0 | 0 | 0.00 | 21.7 | 7.3 | 10.2 |
| 24+00 | 2.00 | 1.72 | 1.33 | 1.20 | 13.56 | 11.60 | 85.5 | 14.5 | 0.17 | 28.8 | 12.9 | 21.5 |
| 24+50 | 1.98 | 1.63 | 1.32 | 1.16 | 13.42 | 10.95 | 81.6 | 18.4 | 0.23 | 30.5 | 14.5 | 25.0 |
| 25+00 | 2.08 | 1.49 | 1.36 | 1.09 | 14.12 | 9.97 | 70.6 | 29.4 | 0.42 | 33.5 | 17.4 | 31.9 |
| 25+50 | 1.90 | 1.42 | 1.28 | 1.05 | 12.86 | 9.52 | 74.0 | 26.0 | 0.35 | 35.1 | 19.2 | 36.6 |
| 26+00 | 1.98 | 1.39 | 1.32 | 1.03 | 13.42 | 9.30 | 69.4 | 30.6 | 0.44 | 35.9 | 20.0 | 38.8 |
| 26+50 | 2.18 | 1.46 | 1.40 | 1.07 | 14.82 | 9.80 | 66.1 | 33.9 | 0.51 | 34.1 | 18.2 | 34.0 |
| 27+00 | 2.00 | 1.59 | 1.33 | 1.16 | 13.56 | 10.70 | 78.9 | 21.1 | 0.27 | 31.2 | 15.1 | 26.0 |
| 27+50 | 2.08 | 1.72 | 1.36 | 1.20 | 14.12 | 11.60 | 82.2 | 17.8 | 0.22 | 28.7 | 12.8 | 21.3 |
| 28+00 | 2.13 | 1.87 | 1.38 | 1.27 | 14.47 | 12.65 | 87.4 | 12.6 | 0.14 | 26.4 | 10.8 | 17.0 |
| 28+50 | 2.25 | 1.87 | 1.43 | 1.27 | 15.31 | 12.65 | 82.6 | 17.4 | 0.21 | 26.4 | 10.8 | 17.0 |
| 29+00 | 2.15 | 1.87 | 1.39 | 1.27 | 14.61 | 12.65 | 86.6 | 13.4 | 0.16 | 26.4 | 10.8 | 17.0 |
| 29+50 | 2.20 | 1.87 | 1.41 | 1.27 | 14.96 | 12.65 | 84.6 | 15.4 | 0.18 | 26.4 | 10.8 | 17.0 |
| 30+00 | 2.22 | 1.90 | 1.42 | 1.28 | 15.10 | 12.85 | 85.1 | 14.9 | 0.17 | 26.0 | 10.5 | 17.0 |
| 30+50 | 2.20 | 1.90 | 1.41 | 1.28 | 14.96 | 12.85 | 85.9 | 14.1 | 0.16 | 26.0 | 10.5 | 16.4 |
| 31+00 | 2.30 | 1.94 | 1.45 | 1.30 | 15.66 | 13.10 | 83.6 | 16.4 | 0.20 | 25.5 | 10.1 | 16.4 |
| 31+50 | 2.25 | 1.94 | 1.43 | 1.30 | 15.31 | 13.10 | 85.6 | 14.4 | 0.17 | 25.5 | 10.1 | 15.5 |
| 32+00 | 2.25 | 1.94 | 1.43 | 1.30 | 15.31 | 13.10 | 85.6 | 14.4 | 0.17 | 25.5 | 10.1 | 15.5 |
| 32+50 | 2.12 | 1.78 | 1.38 | 1.23 | 14.40 | 12.00 | 83.3 | 16.7 | 0.20 | 27.9 | 12.1 | 19.7 |
| 33+00 | 2.06 | 1.71 | 1.35 | 1.19 | 14.02 | 11.50 | 82.1 | 17.9 | 0.22 | 29.0 | 13.1 | 22.1 |
| 33+50 | 2.15 | 1.79 | 1.39 | 1.23 | 14.61 | 12.10 | 82.8 | 17.2 | 0.21 | 27.6 | 11.8 | 19.2 |
| 34+00 | 2.20 | 1.89 | 1.41 | 1.28 | 14.96 | 12.75 | 85.2 | 14.8 | 0.17 | 26.2 | 10.7 | 16.7 |
| 34+50 | 2.12 | 1.90 | 1.38 | 1.28 | 14.40 | 12.85 | 89.2 | 10.8 | 0.12 | 26.0 | 10.5 | 16.4 |
| 35+00 | 2.08 | 1.70 | 1.36 | 1.19 | 14.12 | 11.45 | 81.1 | 18.9 | 0.23 | 29.2 | 13.3 | 17.9 |
| 35+35 | 1.97 | 1.59 | 1.31 | 1.14 | 13.35 | 10.70 | 80.1 | 19.9 | 0.25 | 31.3 | 15.2 | 26.7 |

[a] Velocity head, $H_v$, corresponding to the observed velocity. [b] Width changes from 10.0 ft to 7.0 ft between Station 23+00 and Station 23+50.

that the velocity varied on the same course between several tests. Hence, it is doubtful if this method yields much greater accuracy than the color timing method.

Observations of velocity were recorded for ten flows, ranging from 89 cu ft per sec to 1,005 cu ft per sec.

A cross wave developed at the head of the chute, due to entrance conditions, and was reflected from side-wall to side-wall, but this had practically disappeared before the first point gage station had been reached. On the convex vertical curve the flow left the bottom of the chute and caused a spray to overtop the side-walls. This was particularly true of the larger flows. This spray prevented point gage readings at the first station on the steep slope for all but

TABLE 9.—HYDRAULIC PROPERTIES, KITTITAS CHUTE

| $Q$, cu ft per sec | Station | Hydraulic Properties and Percentage of Air in Water | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Depths (Ft) | | Hydraulic Radius (Ft) | | Area (Sq Ft) | | Percentages | | Ratio, Col. 8 / Col. 7 | Observed velocity $V_o$ | Velocity head[a] $H_v$ | $\frac{V^2}{g R_c}$ |
| | | Observed $y_o$ | Computed $y_c$ | Observed $R_o$ | Computed $R_c$ | Observed $A_o$ | Computed $A_c$ | Water ($\rho$) | Air ($1-\rho$) | | | | |
| | | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) |
| | | (a) Slope 10° 12′ | | | | | | | | | | | |
| 89 | 1–3 | 0.49 | 0.38 | 0.44 | 0.35 | 3.88 | 3.11 | 79 | 21 | 0.27 | 28.6 | 12.72 | 72.6 |
| | 3–5 | 0.50 | 0.41 | 0.45 | 0.37 | 3.96 | 3.25 | 82 | 18 | 0.22 | 27.4 | 11.67 | 63.1 |
| 193 | 1–3 | 0.80 | 0.79 | 0.68 | 0.66 | 6.35 | 6.33 | 99 | 1 | 0.01 | 30.5 | 14.63 | 44.3 |
| | 3–5 | 0.81 | 0.81 | 0.68 | 0.68 | 6.44 | 6.48 | 100 | 0 | 0.00 | 29.8 | 13.81 | 40.6 |
| | 5–7 | 0.83 | 0.79 | 0.70 | 0.66 | 6.55 | 6.33 | 97 | 3 | 0.03 | 30.5 | 14.46 | 43.1 |
| | 5–9 | 0.82 | 0.76 | 0.69 | 0.64 | 6.48 | 6.11 | 94 | 6 | 0.06 | 31.6 | 15.53 | 48.5 |
| 362 | 1–3 | 1.25 | 1.12 | 0.96 | 0.87 | 9.90 | 9.00 | 91 | 9 | 0.40 | 40.2 | 25.13 | 57.8 |
| | 3–5 | 1.25 | 1.07 | 0.96 | 0.84 | 9.95 | 8.54 | 86 | 14 | 0.16 | 42.4 | 27.95 | 66.5 |
| | 5–8 | 1.24 | 1.14 | 0.96 | 0.89 | 9.85 | 9.10 | 92 | 8 | 0.09 | 39.8 | 24.63 | 55.3 |
| | 8–9 | 1.24 | 0.91 | 0.96 | 0.74 | 9.84 | 7.31 | 74 | 26 | 0.35 | 49.5 | 38.1 | 103.0 |
| 401 | 1–3 | 1.26 | 1.26 | 0.98 | 0.98 | 10.06 | 10.66 | 100 | 0 | 0.00 | 37.6 | 21.98 | 44.8 |
| | 3–5 | 1.26 | 1.07 | 0.97 | 0.86 | 9.98 | 9.52 | 95 | 5 | 0.05 | 42.1 | 27.56 | 64.1 |
| 491 | 1–3 | 1.54 | 1.48 | 1.14 | 1.075 | 12.37 | 11.66 | 94 | 6 | 0.06 | 42.1 | 27.56 | 51.3 |
| | 3–5 | 1.55 | 1.48 | 1.13 | 1.075 | 12.30 | 11.69 | 96 | 4 | 0.04 | 42.0 | 27.42 | 51.1 |
| | 5–8 | 1.56 | 1.38 | 1.14 | 1.02 | 12.36 | 11.01 | 89 | 11 | 0.12 | 44.6 | 30.93 | 60.6 |
| | 5–9 | 1.56 | 1.33 | 1.15 | 0.995 | 12.43 | 10.63 | 86 | 14 | 0.16 | 46.2 | 33.19 | 66.7 |
| 587 | 1–3 | 1.80 | 1.65 | 1.26 | 1.165 | 14.32 | 13.19 | 92 | 8 | 0.09 | 44.5 | 30.79 | 56.4 |
| | 3–5 | 1.80 | 1.61 | 1.26 | 1.145 | 14.33 | 13.98 | 91 | 9 | 0.10 | 45.6 | 32.33 | 53.2 |
| | 5–7 | 1.79 | 1.55 | 1.26 | 1.11 | 14.28 | 12.41 | 87 | 13 | 0.15 | 47.3 | 34.78 | 62.7 |
| | 7–9 | 1.80 | 1.42 | 1.26 | 1.04 | 14.36 | 11.33 | 93 | 7 | 0.08 | 51.8 | 41.7 | 80.1 |
| 719 | 1–3 | 2.02 | 1.93 | 1.36 | 1.30 | 16.11 | 15.43 | 96 | 4 | 0.04 | 46.6 | 33.76 | 51.9 |
| | 3–5 | 2.03 | 1.95 | 1.37 | 1.31 | 16.12 | 15.63 | 97 | 3 | 0.03 | 46.0 | 32.90 | 50.2 |
| | 5–7 | 2.05 | 1.82 | 1.38 | 1.25 | 16.33 | 14.55 | 89 | 11 | 0.12 | 49.4 | 37.94 | 57.9 |
| | 7–9 | 2.07 | 1.67 | 1.39 | 1.175 | 16.49 | 13.36 | 81 | 19 | 0.23 | 53.8 | 45.00 | 76.6 |
| 777 | 1–3 | 2.24 | 1.86 | 1.45 | 1.26 | 17.87 | 14.91 | 83 | 17 | 0.21 | 52.1 | 42.2 | 67.0 |
| | 3–5 | 2.22 | 1.90 | 1.45 | 1.29 | 17.68 | 15.24 | 87 | 13 | 0.15 | 51.0 | 40.4 | 62.6 |
| | 5–7 | 2.21 | 1.91 | 1.44 | 1.29 | 17.58 | 15.27 | 87 | 13 | 0.15 | 50.9 | 40.3 | 62.5 |
| | 7–9 | 2.26 | 1.89 | 1.46 | 1.28 | 17.97 | 15.12 | 84 | 16 | 0.19 | 51.4 | 41.1 | 64.2 |
| 922 | 1–3 | 2.52 | 2.14 | 1.56 | 1.40 | 20.02 | 17.14 | 86 | 14 | 0.16 | 53.8 | 45.0 | 64.3 |
| | 3–5 | 2.49 | 2.14 | 1.55 | 1.40 | 19.79 | 17.14 | 87 | 13 | 0.15 | 53.8 | 45.0 | 64.3 |
| | 5–7 | 2.50 | 2.26 | 1.55 | 1.44 | 19.84 | 18.08 | 92 | 8 | 0.09 | 51.0 | 40.4 | 56.1 |
| | 7–9 | 2.51 | 2.13 | 1.56 | 1.39 | 19.96 | 17.04 | 86 | 14 | 0.16 | 54.1 | 45.5 | 65.5 |
| 1,005 | 1–3 | 2.70 | 2.13 | 1.63 | 1.39 | 21.50 | 17.03 | 79 | 21 | 0.27 | 59.0 | 54.1 | 77.8 |
| | 3–5 | 2.65 | 2.31 | 1.62 | 1.46 | 21.11 | 18.49 | 88 | 12 | 0.14 | 54.4 | 46.0 | 63.0 |
| | 5–7 | 2.66 | 2.25 | 1.62 | 1.44 | 21.15 | 18.01 | 86 | 14 | 0.16 | 55.8 | 48.4 | 67.2 |
| | 7–9 | 2.66 | 2.37 | 1.62 | 1.49 | 21.19 | 18.93 | 84 | 16 | 0.19 | 53.1 | 43.8 | 58.8 |

TABLE 9.—(*Continued*)

| $Q$, cu ft per sec | Station | Hydraulic Properties and Percentage of Air in Water: Depths (Ft) Observed $y_o$ (1) | Depths (Ft) Computed $y_c$ (2) | Hydraulic Radius (Ft) Observed $R_o$ (3) | Hydraulic Radius (Ft) Computed $R_c$ (4) | Area (Sq Ft) Observed $A_o$ (5) | Area (Sq Ft) Computed $A_c$ (6) | Percentages Water ($\rho$) (7) | Percentages Air ($1-\rho$) (8) | Ratio, Col. 8 / Col. 7 (9) | Observed velocity $V_o$ (10) | Velocity head[a] $H_v$ (11) | $\frac{V^2}{g R_e}$ (12) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | (*b*) Slope 33° 10′ | | | | | | | | |
| 193 | 12–14 | 1.02 | 0.58 | 0.83 | 0.50 | 8.07 | 4.34 | 55 | 45 | 0.82 | 44.0 | 30.10 | 120.0 |
| | 14–16 | 1.14 | 0.45 | 0.90 | 0.40 | 9.04 | 3.61 | 40 | 60 | 1.50 | 53.4 | 44.3 | 222.0 |
| | 14–15 | 1.14 | 0.45 | 0.90 | 0.40 | 9.04 | 3.57 | 40 | 60 | 1.50 | 54.0 | 45.3 | 226.0 |
| 362 | 11–14 | 1.50 | 0.77 | 1.11 | 0.65 | 11.83 | 6.13 | 52 | 48 | 0.92 | 59.1 | 54.3 | 167.0 |
| | 12–14 | 1.50 | 0.79 | 1.11 | 0.67 | 11.83 | 6.33 | 53 | 47 | 0.89 | 57.2 | 50.9 | 152.0 |
| | 14–15 | 1.63 | 0.80 | 1.17 | 0.67 | 12.88 | 6.41 | 50 | 50 | 1.00 | 56.5 | 49.6 | 148.0 |
| | 14–16 | 1.63 | 0.86 | 1.17 | 0.705 | 12.88 | 6.87 | 53 | 47 | 0.89 | 52.7 | 43.2 | 123.0 |
| 401 | 11–13 | 1.49 | 0.84 | 1.11 | 0.70 | 11.75 | 6.68 | 57 | 43 | 0.75 | 60.0 | 56.0 | 160.0 |
| | 13–16 | 1.51 | 0.80 | 1.12 | 0.67 | 12.08 | 6.43 | 52 | 48 | 0.92 | 62.4 | 60.5 | 181.0 |
| 491 | 12–13 | 1.94 | 1.04 | 1.32 | 0.83 | 15.32 | 8.32 | 55 | 45 | 0.82 | 59.0 | 54.1 | 130.0 |
| | 11–13 | 1.94 | 0.96 | 1.32 | 0.77 | 15.32 | 7.70 | 51 | 49 | 0.96 | 63.8 | 63.5 | 164.0 |
| | 12–14 | 1.94 | 0.99 | 1.32 | 0.80 | 15.32 | 7.91 | 55 | 45 | 0.82 | 62.1 | 60.0 | 150.0 |
| | 13–15 | 1.90 | 0.93 | 1.30 | 0.75 | 15.02 | 7.46 | 50 | 50 | 1.00 | 65.8 | 67.3 | 179.0 |
| | 14–16 | 1.90 | 1.00 | 1.30 | 0.80 | 15.02 | 7.98 | 53 | 47 | 0.89 | 61.5 | 58.8 | 147.0 |
| 587 | 11–13 | 2.05[b] | 1.15 | 1.37 | 0.89 | 16.23[b] | 9.17 | 57 | 43 | 0.75 | 64.0 | 63.4 | 142.0 |
| | 13–15 | 2.05 | 1.10 | 1.37 | 0.86 | 16.16 | 8.80 | 55 | 45 | 0.82 | 66.7 | 69.2 | 161.0 |
| | 13–16 | 2.05 | 1.09 | 1.37 | 0.86 | 16.16 | 8.70 | 53 | 47 | 0.89 | 67.5 | 70.8 | 165.0 |
| 719 | 11–13 | 2.18[b] | 1.21 | 1.43 | 0.925 | 17.26[b] | 9.70 | 56 | 44 | 0.79 | 74.1 | 85.4 | 185.0 |
| | 13–15 | 2.20 | 1.36 | 1.43 | 1.01 | 17.42 | 10.84 | 62 | 38 | 0.61 | 66.3 | 68.3 | 135.0 |
| | 13–16 | 2.20 | 1.28 | 1.43 | 0.97 | 17.42 | 10.24 | 58 | 42 | 0.72 | 70.2 | 76.7 | 158.0 |
| 777 | 11–13 | 2.35[b] | 1.47 | 1.50 | 1.07 | 18.61[b] | 11.79 | 63 | 37 | 0.59 | 65.9 | 67.5 | 126.0 |
| | 13–16 | 2.40 | 1.08 | 1.51 | 0.85 | 19.00 | 8.64 | 46 | 54 | 1.17 | 89.9 | 125.6 | 296.0 |
| 922 | 12–13 | 2.63[b] | 1.58 | 1.60 | 1.13 | 20.83[b] | 12.61 | 61 | 39 | 0.64 | 73.1 | 83.1 | 147.0 |
| | 12–14 | 2.63[b] | 1.63 | 1.60 | 1.15 | 20.83[b] | 13.08 | 63 | 37 | 0.59 | 70.5 | 77.3 | 134.0 |
| | 13–15 | 2.68 | 1.54 | 1.61 | 1.11 | 21.28 | 12.29 | 58 | 42 | 0.72 | 75.0 | 87.5 | 158.0 |
| | 14–16 | 2.68 | 1.68 | 1.61 | 1.18 | 21.28 | 13.44 | 63 | 37 | 0.59 | 68.6 | 73.1 | 124.0 |
| 1,005 | 12–15 | .... | 1.51 | .... | 1.09 | ..... | 12.09 | .. | .. | .... | 83.1 | 107.3 | 197.0 |
| | 13–15 | .... | 1.53 | .... | 1.10 | ..... | 12.22 | .. | .. | .... | 82.2 | 105.0 | 191.0 |
| | 14–16 | 2.94[b] | 1.69 | 1.71 | 1.185 | 23.20[b] | 13.56 | 58 | 42 | 0.72 | 74.1 | 85.4 | 144.0 |
| | | | | | (*c*) Slope 0° 04′ | | | | | | | | |
| 362 | 18–20 | 1.52 | 1.31 | 1.12 | 0.98 | 12.03 | 10.49 | 87 | 13 | 0.15 | 34.5 | 18.50 | 37.7 |
| 491 | 17–20 | 1.82 | 1.14 | 1.06 | 0.89 | 14.41 | 9.13 | 63 | 37 | 0.59 | 53.8 | 45.0 | 101.0 |
| | 18–20 | 1.82 | 1.35 | 1.06 | 1.01 | 14.41 | 10.81 | 74 | 26 | 0.35 | 45.4 | 32.05 | 63.5 |
| 587 | 17–19 | 2.14 | 2.18 | 1.41 | 1.40 | 16.95 | 17.42 | 100 | 0 | 0.00 | 33.7 | 17.67 | 25.2 |
| | 17–20 | 2.14 | 1.50 | 1.41 | 1.09 | 16.95 | 12.00 | 71 | 29 | 0.41 | 48.9 | 37.18 | 68.2 |
| 719 | 17–19 | 2.36 | 1.56 | 1.50 | 1.11 | 18.69 | 12.50 | 67 | 33 | 0.49 | 57.5 | 51.4 | 92.6 |
| | 17–20 | 2.36 | 1.41 | 1.50 | 1.14 | 18.69 | 11.27 | 61 | 39 | 0.64 | 63.8 | 63.3 | 111.0 |
| 777 | 17–19 | 2.61 | 1.80 | 1.58 | 1.24 | 20.76 | 14.36 | 70 | 30 | 0.43 | 54.1 | 45.5 | 73.3 |
| 922 | 17–20 | 2.91 | 1.97 | 1.68 | 1.32 | 22.93 | 15.73 | 68 | 32 | 0.47 | 58.6 | 53.4 | 80.9 |
| | 18–20 | 2.91 | 2.06 | 1.68 | 1.35 | 22.93 | 16.46 | 71 | 29 | 0.41 | 56.0 | 48.8 | 72.3 |
| 1,005 | 17–20 | 2.90 | 2.02 | 1.68 | 1.34 | 22.98 | 16.16 | 71 | 29 | 0.41 | 62.2 | 60.2 | 89.9 |
| | 18–20 | 2.90 | 2.27 | 1.68 | 1.45 | 22.98 | 18.17 | 80 | 20 | 0.25 | 55.3 | 47.6 | 65.7 |

[a] Velocity head, $H_v$, corresponding to the observed velocity. [b] Point gage readings made at the end of the section because of spray.

TABLE 10.—TRAJECTORY OF WATER SURFACE AT CONVEX VERTICAL CURVE, HAT CREEK CHUTE

(395 Cu Ft per Sec)

| Station | Mean velocity (ft per sec) | Depths, in Feet | | | | Elevations, in Feet | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Mean | | Maximum | | Bottom | Surface | | Maximum Computed | | |
| | | Observed | Corrected[a] | Observed | Corrected[a] | | Mean | Maximum | 76 ft per sec | 84 ft per sec | 91 ft per sec |
| $m$ | .... | .... | .... | .... | .... | ..... | ..... | ..... | 302.11 | 369.04 | 433.09 |
| $y_t$ | .... | .... | .... | .... | .... | ..... | ..... | ..... | 14.28 | 17.44 | 20.46 |
| $x_t$ | .... | .... | .... | .... | .... | ..... | ..... | ..... | 65.68 | 80.23 | 94.13 |
| 2+75 | 75.9[b] | 1.85 | 1.73 | 2.15 | 2.11 | 2,853.7 | 2,855.4 | 2,855.8 | 2,855.8 | 2,855.8 | 2,855.8 |
| 3+00 | 76.0 | 3.52 | 2.92 | 4.2 | 3.49 | 2,841.9 | 2,844.8 | 2,845.4 | 2,844.6 | 2,844.9 | 2,845.1 |
| 3+25 | 75.8 | 4.60 | 3.82 | 6.0 | 4.98 | 2,827.9 | 2,831.7 | 2,832.9 | 2,831.2 | 2,832.3 | 2,833.0 |
| 3+50 | 74.9 | 3.50 | 2.90 | 4.9 | 4.06 | 2,813.4 | 2,816.3 | 2,817.5 | 2,814.8 | 2,817.6 | 2,819.2 |
| 3+75 | 72.3 | 1.97 | 1.70 | 2.2 | 1.91 | 2,800.5 | 2,802.2 | 2,802.4 | 2,795.4 | 2,799.7 | 2,802.6 |

[a] Corrected for slope. [b] Total drop at Station 2+75 is 129.3 ft. The theoretical velocity equals 91.3 ft per sec.

the low flow; and even at the second station readings were prevented on the highest flows.

From the data furnished by the U. S. Bureau of Reclamation, calculations of the retardation factor have been made by the three methods previously described from observations for the 10° 12′ slope and for the 33° 10′ slope. Velocity observations were not taken on the vertical curves or in the taper section, which prevents the calculation of a complete energy profile for this chute.

TABLE 11.—SUPERELEVATION OF WATER SURFACE, SOUTH CANAL CHUTE; ANOTHER PROJECT

(Radius of Curve, 142 Ft)

| Station | Observed velocity $V_o$ | Tan $\psi$ | Gage Readings | | Difference in elevation (ft) | Slope[a] in a 7-ft width (tan $\psi$) |
|---|---|---|---|---|---|---|
| | | | Left side | Right side | | |
| 23+00 | 16.6 | 0.060 | 1.88 | 2.75 | 0.87 | 0.124 |
| 23+25 | 19.2 | 0.081 | 2.15 | 2.40 | 0.25 | 0.036 |
| 23+50 | 21.7 | 0.103 | 1.60 | 2.90 | 1.30 | 0.186 |
| 23+75 | 25.2 | 0.139 | 1.70 | 2.60 | 0.90 | 0.129 |
| 24+00 | 28.8 | 0.181 | 1.40 | 2.60 | 1.20 | 0.172 |
| 24+25 | 29.6 | 0.192 | 1.40 | 2.55 | 1.15 | 0.164 |
| 24+50 | 30.5 | 0.204 | 1.20 | 2.75 | 1.55 | 0.222 |
| 24+75 | 32.0 | 0.224 | 1.20 | 2.50 | 1.30 | 0.186 |
| 25+00 | 33.5 | 0.245 | 0.70 | 3.45 | 2.75 | 0.394 |
| 25+25 | 34.3 | 0.257 | 0.70 | 3.35 | 2.65 | 0.379 |
| 25+50 | 35.1 | 0.269 | 0.90 | 2.90 | 2.00 | 0.286 |

[a] The computed tangent agrees reasonably well with the observed tangent. The superelevation of Station 23+00 or the P.C. of the curve is due to cross waves resulting from the curvature of the channel upstream.

The hydraulic properties of the water prism, together with percentage of air in the water and the values of the parameter $\frac{V^2}{g R_c}$, were tabulated for each of the observations in Table 9.

### ANALYSES OF RESULTS

The values of the retardation factor conform, in general, with the range discussed under the heading, "Theory of Flow in Chutes." Exceptions were noted near the inlets where high values were recorded as a result of entrance conditions. Similar high values occurred

below the convex vertical curve on the Hat Creek chute due to the spreading of the cross section as the water left the bottom of the chute. Also, on the sharp horizontal curve in the South Canal chute an increased friction loss was noted. In each chute the most representative values in both the Manning and Kutter formulas have been tabulated, leaving the exceptions for separate comment.

The retardation factors obtained from the observed areas and observed velocities are equivalent to those observed in normal channels. This value, therefore, has been designated as $n$. The retardation factor obtained by using the observed velocity and the area computed by $\frac{Q}{V_o}$ is designated as $n_c$, and the values obtained by using the observed areas and the velocity computed by $\frac{Q}{A_o}$ has been designated as $n_o$. The use of observed velocities and observed areas gives results for the retardation factor comparable with those obtained in wood and concrete channels under normal flow conditions as contrasted with the shooting flow being tested. The values of $n_c$ are less than normal values and correspond to those which could be computed by Eq. 26. The value of $n_o$ is considerably larger than the results obtained by the first two methods.

The results obtained for $n_o$ on all chutes (see Table 12) are the most in-

TABLE 12.—COMPARISON OF RETARDATION FACTORS

| Formula | HAT CREEK | | RAPID FLUME | | SO. CANAL CHUTE | | KITTITAS CHUTE | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | Slope, 10° 12′ | | Slope, 33° 10′ | |
| | From: | To: | From: | To: | From: | To: | From: | To: | From: | To: |
| $n$— | | | | | | | | | | |
| Manning | 0.010 | 0.013 | 0.010 | 0.012 | 0.012 | 0.013 | 0.015 | 0.016 | 0.019 | 0.021 |
| Kutter | 0.010 | 0.013 | 0.011 | 0.012 | 0.012 | 0.013 | 0.015 | 0.016 | 0.019 | 0.021 |
| $n_c$— | | | | | | | | | | |
| Manning | 0.008 | 0.010 | 0.008 | 0.010 | 0.011 | 0.012 | 0.014 | 0.015 | 0.014 | 0.015 |
| Kutter | 0.008 | 0.010 | 0.008 | 0.010 | 0.011 | 0.012 | 0.014 | 0.015 | 0.014 | 0.015 |
| $n_o$— | | | | | | | | | | |
| Manning | 0.024 | 0.031 | 0.014 | 0.019 | 0.015 | 0.016 | 0.016 | 0.018 | 0.030 | 0.055 |
| Kutter | 0.023 | 0.030 | 0.014 | 0.018 | 0.015 | 0.016 | 0.016 | 0.018 | 0.030 | 0.045 |

consistent of the three methods used, largely due to the vagaries of the computed energy gradient. Therefore, this method has little to commend it either in theory or in practice.

The values for the Hat Creek chute include the tests for 115, 163, 368, and 395 cu ft per sec as observed on the straight section of the channel between Stations 0+50 and 2+75. The retardation factor materially increased on the steep grade below the convex vertical curve and in the first portion of the bucket. Using the observed values of velocity and area, the values of Kutter's and Manning's $n$ average about 0.020 in this section of the channel. The values of $n_c$ and $n_o$ correspondingly increase over values previously cited by the

other methods of computation. The increase in the retardation factor in this section of the channel is due to the increased friction resulting from the path taken by the trajectory of the flow around the vertical curve. This trajectory has been calculated for $Q = 395$ cu ft per sec by means of Eq. 31, using in the calculations the mean velocity and also velocities 10% and 20% above the mean velocity. These results are given in Table 10.

The mean velocity at the start of the vertical curve was 76 ft per sec, but a comparison of the results in Table 10 indicates that the water surface follows a trajectory corresponding to a velocity of 91 ft per sec. This is equal to the theoretical velocity of the water falling from the inlet of the chute to the start of the vertical curve. In a short channel the water surface in the center of the chute, therefore, is not retarded by friction, but travels at the theoretical velocity. In compensation, the velocity at the side and bottom of the incline necessarily must travel at less than the mean velocity, with the result that on vertical curves the bottom layers should adhere to the floor of the chute, whereas the upper layers separate from the water prism, throwing the stream into a spray. Air resistance ultimately slows the surface velocity as a comparison of the calculations with the observations shows that the surface curve drops suddenly at Station 3+25 after following the trajectory for a velocity of 91 ft per sec for a distance of 50 ft. The increase in the retardation factor in this section of the chute indicates, as previously mentioned, that there exists a possibility of dissipating energy by this means. The same principle is utilized to some extent in discharging water from a chute bucket above the water surface of the discharge pool. The throwing of the jet into a spray assists in dissipating the energy, in addition to a further dissipation of energy by eddies and turbulence in the pool itself.

The results on Rapid Flume correspond in a large measure to those obtained from the tests on the Hat Creek chute. In this case the convex vertical curve afforded a uniform retardation factor throughout the length of the chute after the turbulent conditions at the inlet had been passed.

On the South Canal chute the computed retardation factors are more erratic. Some of the apparent variation is probably due to faulty observations of area or velocities. The average results for the entire length of channel are reasonable, although when the roughness of the bottom is considered, a value of $n$ between 0.014 and 0.015 would have been expected, rather than the lower value actually obtained.

On the steep section of the chute constructed on the 142-ft radius, located between Station 23+00 and Station 25+00, the value of $n_o$ averaged 0.021. This was due to the fact that the increased insufflation of air in this portion of the channel increased the cross section and correspondingly reduced the computed values of the velocity.

The superelevation of the water surface around the 142-ft radius curve has been tabulated and compared with the theoretical values computed according to Eq. 32. These results are given in Table 11. The agreement between the actual and theoretical results is sufficiently close to indicate that it is evident that the flow around this curve would have been improved by the superelevation of the bottom of the channel in accordance with the theoretical values.

The flow, thereby, would have been kept within the channel walls and the overtopping of the outer wall by spray would have been avoided. At the other curves along the channel the radius of curvature was not sufficiently sharp to cause overtopping of the wall. Hence, the necessity for superelevation of the bottom is not so essential.

On the Kittitas chute the slope of the energy gradient could be determined only from velocity observations taken over relatively short detached portions of the chute. The energy gradient computed from these data proved to be erratic, but as the average of the results approximated the bottom slope of the chute, this slope was used for the energy gradient. The values of the retardation factor are larger than those obtained on the tests of the other three chutes, but the difference may be due in part to the fact that the slope of the energy gradient is actually less than the bottom slopes, or from unusual roughness of the bottom of the channel not apparent from the information available. The high value of $n$ on the 33° 10′ slope based on observed areas and velocities is undoubtedly due in part to the stream having been thrown into spray on the convex vertical curve immediately above this incline (see Table 12). The same effect was noted on the test at the Hat Creek chute.

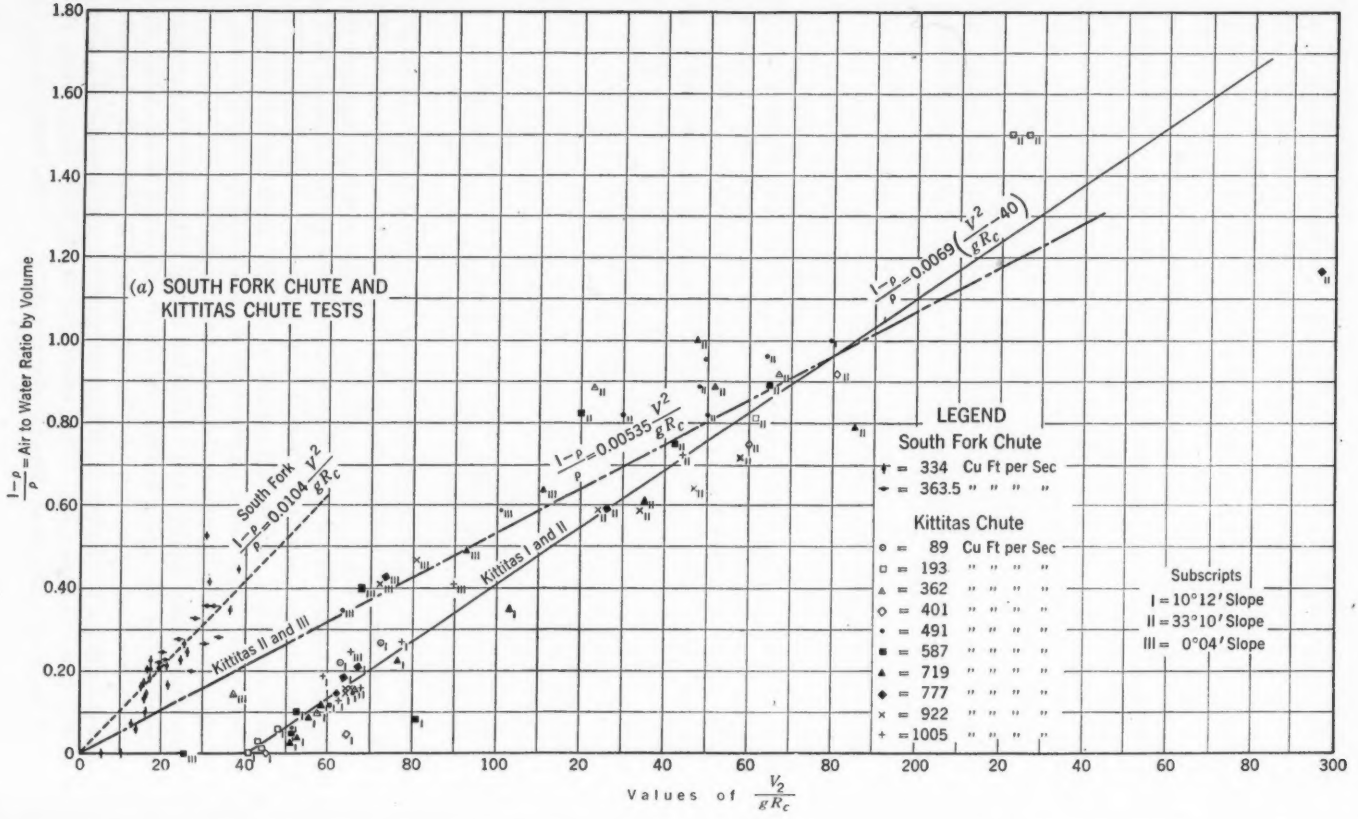
The relation of the ratio of air to water and the parameter $\frac{V^2}{g R_c}$ has been plotted graphically. All of the four tests described have been shown in Fig. 10.

Although there is some scattering of the plotted data, the results are reasonably consistent, considering the difficulty in obtaining the velocity and area measurements. The graphs afford an experimental demonstration of the linear relation between the air-water ratio and the Froude number expressed in Eq. 24*a*. The values of $K$ and $K_1$ for the various chutes are as follows:

| Name of chute | $K$ | $K_1$ |
|---|---|---|
| Hat Creek chute | 0.0041 | 0 |
| Rapid Flume | 0.00355 | 0 |
| South Canal | 0.0104 | 0 |
| Kittitas chute (upper end) | 0.0069 | 0.276 |
| Kittitas chute (lower end) | 0.00535 | 0 |

The values of $K$ are reasonably consistent with the general condition of roughness of the respective channels. In Rapid Flume the wood construction results in the lowest value of the coefficient. Of the concrete channels Hat Creek has the lowest value, although the roughness of the channel in the Kittitas chute (judged from the photographs in the case of the latter) is not so great as than of Hat Creek. However, the results from these two chutes are in close accord. The South Canal chute of another project with its rough bottom and sinuous alinement has the highest value of $K$.

The selection of the proper value of $K$ is left to the judgment of the engineer using the presented data as a guide. In concrete chutes, if the water discharged down the channel is relatively clear, as is the case with spillway chutes from reservoirs or from canals normally carrying clear water, the use of a value of $K$ of 0.006 is recommended. This coefficient is slightly larger than the average of the tests and affords some factor of safety. If the chute serves as


(a) SOUTH FORK CHUTE AND KITTITAS CHUTE TESTS
$\frac{I-\rho}{\rho}$ = Air to Water Ratio by Volume
Values of $\frac{V_2}{gR_c}$
South Fork $\frac{I-\rho}{\rho}=0.0104\frac{V^2}{gR_c}$
$\frac{I-\rho}{\rho}=0.00535\frac{V^2}{gR_c}$
$\frac{I-\rho}{\rho}=0.0069\left(\frac{V^2}{gR_c}-40\right)$
Kittitas II and III
Kittitas I and II
LEGEND
South Fork Chute
= 334 Cu Ft per Sec
= 363.5 " " "
Kittitas Chute
= 89 Cu Ft per Sec
= 193 " " "
= 362 " " "
= 401 " " "
= 491 " " "
= 587 " " "
= 719 " " "
= 777 " " "
= 922 " " "
= 1005 " " "
Subscripts
I = 10°12′ Slope
II = 33°10′ Slope
III = 0°04′ Slope

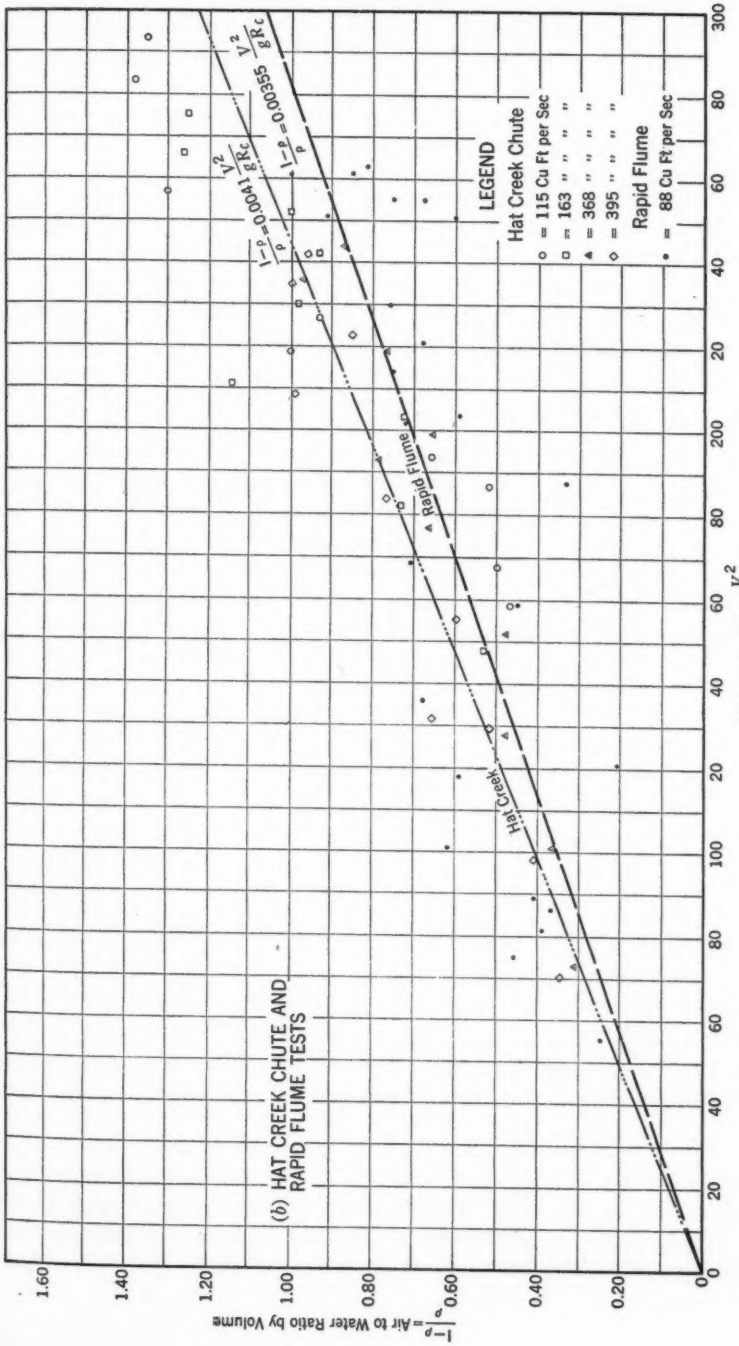

(b) HAT CREEK CHUTE AND RAPID FLUME TESTS
LEGEND
Hat Creek Chute
○ = 115 Cu Ft per Sec
□ = 163 " " "
▲ = 368 " " "
◇ = 395 " " "
Rapid Flume
• = 88 Cu Ft per Sec
$\frac{1-\rho}{\rho}=0.0041\frac{V^2}{gR_c}$
$\frac{1-\rho}{\rho}=0.00355\frac{V^2}{gR_c}$
Hat Creek
Rapid Flume
$\frac{1-\rho}{\rho}$ = Air to Water Ratio by Volume
Values of $\frac{V^2}{gR_c}$
0 20 40 60 80 100 20 40 60 80 200 20 40 60 80 300
0.20 0.40 0.60 0.80 1.00 1.20 1.40 1.60

FIG. 10.—RELATION OF AIR-WATER RATIO TO $\frac{V^2}{gR_c}$

an inclined drop structure to convey water in irrigation canals from bench lands to lower levels, sand carried in the water will roughen the bottom, and the use of higher value of $K$ comparable to that determined from the South Canal chute is more desirable.

More experiments over a wide range of channels would give added confidence to the proper selection of the factor $K$ and $K_1$. The straight-line relation of the air-water ratio to a dimensionless parameter indicates its application to all ranges of velocity and channel sizes from the smallest to the largest.

It will be noted that the application of the formula to large channels will result in a smaller air-water ratio for given velocities than computed in small channels. This conforms with experience in observation of this phenomenon.

One word of caution must be given: Eq. 24*a* will not apply to very short channels in which the distance of travel is too short to entrain air; neither will it apply to streams of such small size that surface tension is an important factor. In general, distance of travel must be 100 ft or more, unless air is entrained from turbulent entrance conditions. Despite these minor limitations the formula should be of value in designing spillway chutes.

A comparison of the actual maximum depths in the chutes with the height of side-walls disclosed that a large factor of safety has been allowed in the design to cover the effect of the little known factors under discussion. On horizontal curves of sharp radius (as, for example, in South Canal chute) or on sharp convex vertical curves as in Hat Creek and Kittitas chutes, the additional allowance proved to be insufficient. However, on straight stretches or on moderate curves, the allowance for freeboard is very generous. This is indicated by a comparison of maximum depth in the chutes, except the aforementioned curved sections, with the height of side-walls. Depths near the inlet before the water had accelerated have also been omitted. The average depths in the remaining portion of the chutes were well below the following maximum values:

| Name of chute | Maximum depth (ft) | Height of walls (ft) |
|---|---|---|
| Hat Creek | 2.5 | 5.0 |
| Rapid Flume | 1.2 | 2.4 |
| South Fork (10-ft width) | 2.0 | 3.5 to 4.5 |
| South Fork (7-ft width) | 3.0 | 3.5 to 5.25 |
| Kittitas | 3.5 | 7.0 |

In most instances a more exact knowledge of the performance to be expected of the water at high velocities would have permitted a reduction in the freeboard allowance with a corresponding economy in the cost of the structure.

## Conclusions

The flow of water on steep gradients is essentially different from that found in channels of ordinary slope. Velocities are greater than the critical, and range from a minimum value of 10 to 15 ft per sec to maximum values of from 80 to 100 ft per sec. The kinetic energy greatly exceeds the static pressure of the water prism.

The initial entrainment of air depends to a great extent on entrance conditions. With a well-designed entrance, the water accelerates uniformly, gradually drawing air into the water prism. This insufflation develops first along the side-walls and spreads gradually into the section until the admixture of air and water covers the entire wetted area. If the water drops over a weir at the inlet to the chute, turbulent conditions with entrained air will exist from the head of the chute throughout the length of the channel. Similar conditions exist when the approach is not in alinement with the chute channel, or when the inlet of the chute is narrowed too rapidly. Under these latter circumstances high, rough, cross waves develop which are reflected from side to side of the chute until the waves are dissipated or the end of the incline is reached.

The flow of water down a steep channel with entrained air does not follow the normal law of hydraulics, as expressed by the continuity equation $Q = A\,V$. The observed area of the stream, before application in the equation, must be reduced by the percentage of air in the admixture. The retardation factors $n$ in the Manning or Kutter formula, obtained by using observed velocities and areas, agree with those obtained in normal channel flow in similar materials. This relation is demonstrated to be correct, both in theory and from actual observations. For practical purposes computations are simplified in the design of chutes by the use of the computed area, assuming no air entrainment. This necessitates the use of a value of $n$ smaller than that normally used, and one that depends for its value on the velocity of the water, as indicated by Eq. 26.

Bernoulli's equation then can be written, starting from the critical section of the channel at the inlet and extending down the chute to the outlet. For rectangular chutes, increments of velocity are assumed and the length of travel required to gain this increment is calculated by Eq. 17. The resulting velocities are then true water velocities, but such calculations yield wetted areas less than those actually existing. The actual cross section of air plus water in the channel can be determined by Eq. 24*b*, with the selection of a coefficient $K$ based on the data presented in Fig. 10, which give the ratio of air to water for values of the parameter $\frac{V^2}{g\,R_c}$. For channels other than rectangular, Eq. 27 and the procedure of calculation as outlined afford a rational solution of the flow in steep chutes.

Convex vertical curves must not be made sharper than the trajectory of the flowing water falling under the action of gravity as determined from Eq. 31, if the stream is expected to adhere to the bottom of the channel. The portion of the water prism near the surface and in the center of the channel travels at a velocity from 15% to 20% greater than the mean velocity in the cross section and this maximum velocity must be used in computing trajectories. In chutes of short length the maximum velocity will be found to be equal to the theoretical velocity, resulting from the vertical drop between the inlet and the point of vertical curvature. If the side-walls are of sufficient height to prevent overtopping, no serious consequence results from the jet leaving the bottom of the channel. The evidence indicates that more energy is dissipated when the stream cross section is expanded by leaving the floor of the chute than would

otherwise be the case. Under extreme conditions damage to the concrete floor of the chutes may result from cavitation.

Flow around horizontal curves should be avoided whenever possible. If used, they should be installed at the head of the chute before the water has had opportunity to gain great momentum. On large channels they should never be used unless the performance of the water has been verified by model tests. Model tests are also desirable to check entrance conditions. The superelevations of the water surface on curves correspond with that computed by the centrifugal theory as indicated by Eq. 32. In cases where the channel is too wide with respect to the depth, a large percentage of the water will either follow around the outside of the curve, leaving the inner portion of the floor exposed, or in some instances the slope of the superelevation will not be uniform across the wetted section, due to the variation in velocities across the water prism. On sharp curves, if the computed superelevation of the water surface exceeds a value of $\tan \psi = 0.120$, the superelevation of the bottom is recommended.

More experimental data over a wider range of channel sizes and with varying degrees of roughness would add to the engineering knowledge of the phenomenon of flow at high velocities.

## Acknowledgments

The field work on these tests was done under the direction of the late C. E. Grunsky, Jr., M. Am. Soc. C. E. Acknowledgment is made for the cooperation of I. C. Steele, M. Am. Soc. C. E., and F. R. George of the Pacific Gas and Electric Company in arranging for the tests and for maintenance of uniform flow conditions during the observations. Credit for the collection of the data must be given to those employees of the East Bay Municipal Utility District attached to the Hydrographic Division, who took part in the tests and contributed to their success.

Indebtedness is acknowledged to B. T. Ploeger and R. E. Hall, Jun. Am. Soc. C. E., for valuable assistance in the calculations and the preparation of the drawings. The kindness of the U. S. Bureau of Reclamation in permitting access to the results of their tests on the Kittitas chute of the Yakima Project is appreciated.

The tabulations, figures, and photographs have been condensed in this paper, but the complete original manuscript has been placed on file for reference at Engineering Societies Library.[18]

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# CLOSED CONDUIT FLOW

BY A. A. KALINSKE,[18] ASSOC. M. SOC. C. E., AND JAMES M. ROBERTSON,[19] JUN. AM. SOC. C. E.

SYNOPSIS

The general problem treated in this paper is the study of air entrainment by flowing water in pipes, particularly in reference to the removal of air pockets from water supply lines. The formation of a hydraulic jump in a pipe is an important method by which air can be removed, and this paper presents data relating to the ability of a hydraulic jump to entrain air and have it carried away by the flowing water. Quantitative data are presented relating the characteristics of the jump with the rate of air removal, and also data regarding the hydraulic features of jumps in closed conduits.

## THE GENERAL PROBLEM

When an air pocket is formed at the summit of a pipe line or beyond a partly open gate valve, the conduit flows only partly full, and at the end of the pocket a hydraulic jump usually occurs. The air may be removed, of course, by use of an air valve; but this is not feasible if the pipe line is under sub-atmospheric pressure. In any case, it is of interest to know under what conditions an air valve may be necessary or, if an air valve cannot be or is not used, whether the air can be removed by the flowing water.

Since the hydraulic jump at the end of an air pocket is a common phenomenon, it was decided that the first phase of this investigation should be concerned with the ability of a hydraulic jump to entrain air. This paper is specifically concerned with this particular item. Of course, the entrainment of air by the hydraulic jump is not the sole item to consider in the general problem of air removal from pipe lines; but it is an important feature of the general problem.

The violent eddies of a hydraulic jump at the end of an air pocket will entrain air and pump this air into the conduit beyond the jump. Whether all the air entrained by the jump will be carried along by the flowing water beyond the jump is another problem to be considered. Also the problem of the movement of finite bubbles of air in the pipe line beyond a hydraulic jump must be considered. This may be the entire problem involved if the conduit is at a very flat slope and no jump forms. K. R. Kennison,[20] M. Am. Soc. C. E., has indicated that if the slope of the conduit is not greater than the hydraulic gradient, such bubbles will be moved along without any difficulty. He also discusses the use of the hydraulic jump to remove air from conduits.

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[19] Care, David Taylor Model Basin, Bureau of Ships, U. S. Navy, Washington, D. C.; formerly Research Fellow in Hydraulics, Univ. of Iowa, Iowa City, Iowa.

[20] "The Design of Pipe Lines," by K. R. Kennison, *Journal*, N. E. W. W. A., Vol. 47, 1933, pp. 27–39.

## Experimental Apparatus and Procedure

To study the entrainment of air by a hydraulic jump in circular pipes, a pipe line made of 6-in. transparent "Lucite" was set up in the laboratory. The line was about 35 ft long with an inside diameter, $D$, of 0.49 ft. The line could be placed at any slope up to 30%. Water was discharged into it through a sliding gate, the head on which could be controlled; thus the depth and velocity could be varied independently. Fig. 11 is a photograph of the pipe line at a 30% slope. Note the hydraulic jump and the large volume of air beyond it, indicating the jump is pumping a large quantity of air.

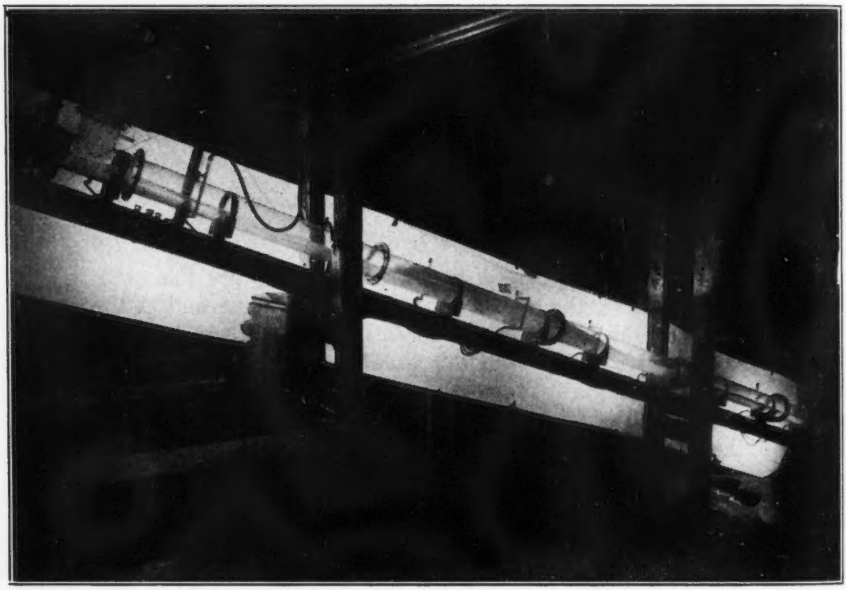


Fig. 11.—General View of 6-In. Transparent Pipe Installation

The rate of air pumped by the jump was measured by means of standard orifices. To maintain steady conditions air was supplied from an independent source to the pocket above the jump at a rate equal to the rate at which the jump was pumping air. At the lower end of the pipe line the water flowed into a rectangular vertical box open at the top to permit the escape of the air. A gate on this box controlled the water level in it and thereby the pressure on the pipe line. The rate of water flow was measured by calibrated orifices.

Studies were made with the line horizontal and at slopes of 0.2%, 2.0%, 5.0%, 10.0%, 20.0%, and 30.0%. For any particular slope various gate openings were used thus determining the depth of flow ahead of the jump. A number of different water discharges were run for each gate opening. Water depth and pressure readings were taken along the pipe line for each run. The water flow approaching the jump was always kept under atmospheric pressure. This

item did not have any influence on the results since a higher or lower pressure could have been maintained simply by raising or lowering the pressure on the entire system.

## Notation

The letter symbols used in this paper are defined where they first appear and are assembled for convenience of reference in the Appendix.

## Results on Air Demand of Jump

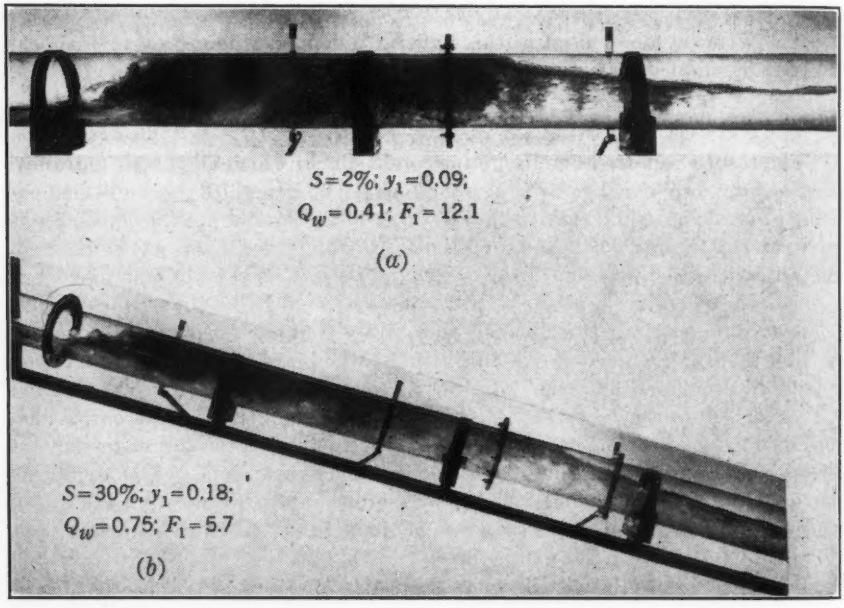
The results reported herein are for conditions in which all the air entrained by the jump is carried out of the pipe line. The general flow conditions obtained are typically illustrated in Fig. 12, where $y_1$ is the initial depth, $Q_w$ is the water discharge, $Q_a$ is the air discharge, and $\mathbf{F}_1$ is the Froude number of the approaching flow defined as $V_1/\sqrt{g\,y_e}$, where $V_1$ is the approach velocity, and $y_e$, the effective depth, is the water area divided by the surface width. The air pumped into the flowing water by the jump forms a large pocket beyond the jump which extends to the point where the air leaves the pipe line. For low water flows and especially at the low slopes a single long pocket did not exist, but rather a series of relatively large bubbles was present, which moved downstream. Where a single long pocket did occur the water flow in this pocket was at the normal depth for that particular slope. The air in this downstream pocket was under higher pressure than the air ahead of the jump, since the pressure head of the lower stage of the jump was always greater than the pipe diameter.

For any given initial depth of flow and slope of pipe, as the water discharge was decreased, a point was reached where the air pumped into the jump exceeded the rate at which air was carried along the pipe beyond the jump. Visually this critical point was quite apparent. The air pumped by the jump would form a large bubble just beyond the jump. This bubble did not extend to the end of the pipe, but some of the air from it would be carried downstream in smaller bubbles. However, periodically this large bubble would blow out over the jump causing the jump to move downstream; then another jump would form and the process would be repeated. It appeared that for discharges below this critical value the average rate of air removal from the pipe line was controlled by the flow conditions below the jump and not by the jump itself.

Considering the rate of air entrainment by a hydraulic jump in a circular pipe, general analyses indicated that for geometrically similar conditions it should depend on the water discharge and the intensity of agitation or turbulence of the jump. The relative intensity or violence of the jump is dependent on the initial Froude number, $\mathbf{F}_1$, which in open channel flow is found to be related to the energy loss in the jump. The slope of the pipe line, $S$, might also affect the air entrainment by the jump; but this did not seem so likely. Dimensional considerations thus lead to the following functional relationship:

$$\frac{Q_a}{Q_w} = \text{function}\left(\mathbf{F}_1, \frac{y_1}{D}, S\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

A plotting of the data obtained did not reveal any particular variation of the air-water ratio with $y_1/D$ or $S$. The only consistent variation was with $\mathbf{F}_1$ (see Fig. 13). The air-water ratio is plotted against $(\mathbf{F}_1 - 1)$, since no



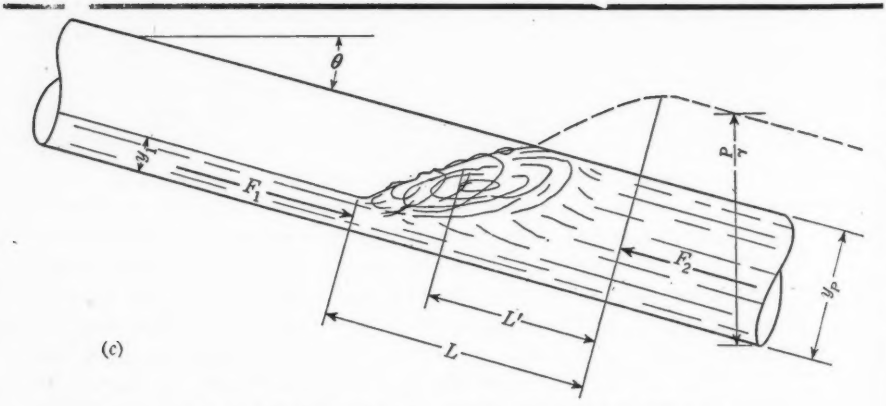




FIG. 12.—ELEMENTS OF HYDRAULIC JUMP IN SLOPING PIPE LINE

hydraulic jump would occur when $\mathbf{F}_1 = 1$. From this it appears that the functional relationship of Eq. 33 can be expressed by the equation

$$\beta = \frac{Q_a}{Q_w} = 0.0066\ (\mathbf{F}_1 - 1)^{1.4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

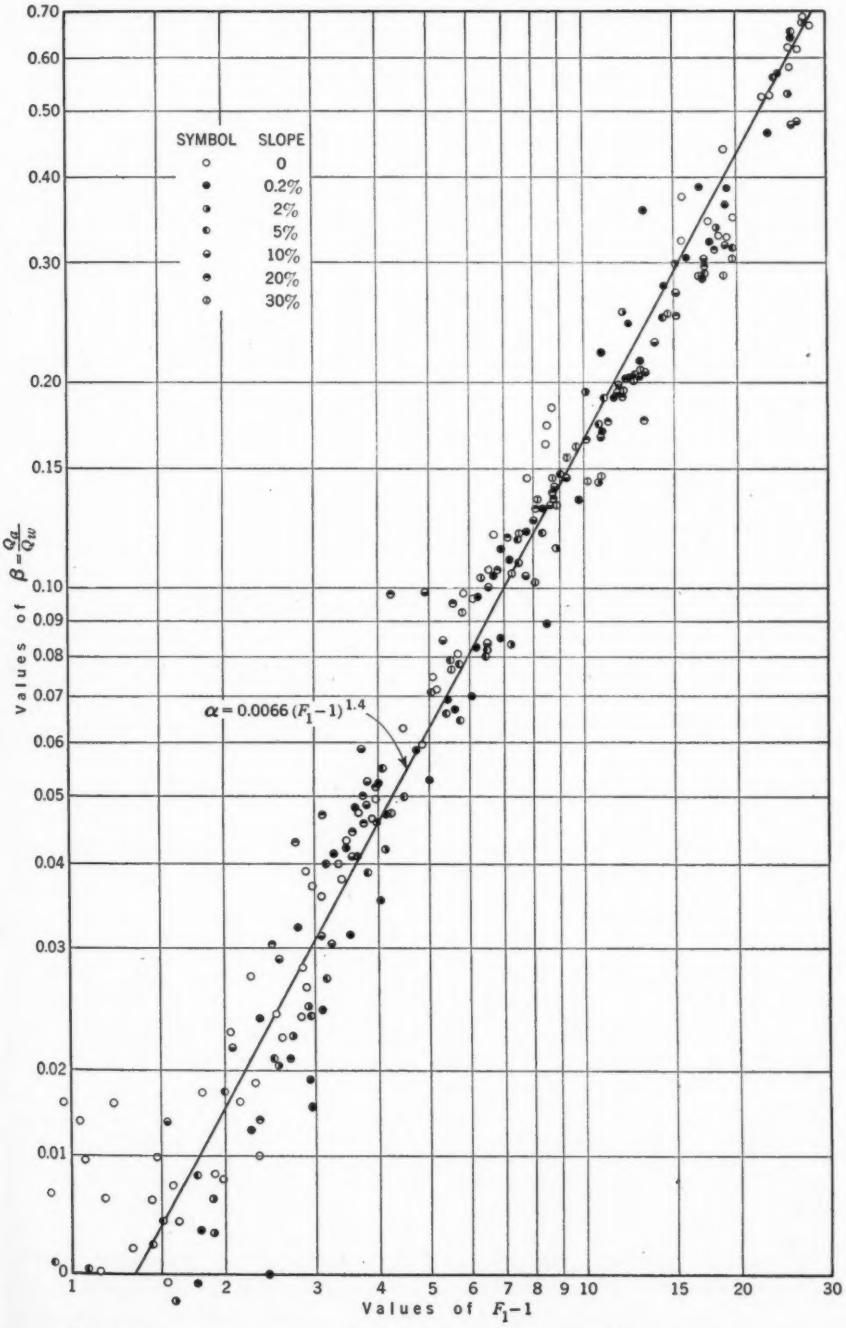



FIG. 13.—CORRELATION OF DATA ON RATE OF AIR ENTRAINMENT BY HYDRAULIC JUMP

It should be emphasized that the data in Fig. 13 are for conditions in which the pipe line carried along and discharged all the air the jump entrained. For any value of $y_1/D$, there was a value of the Froude number $\mathbf{F}_1$ for which the pipe line would carry only a part of the air pumped into the water by the jump. This critical value of $\mathbf{F}_1$ for the various relative depths and slopes studied is shown in Fig. 14.

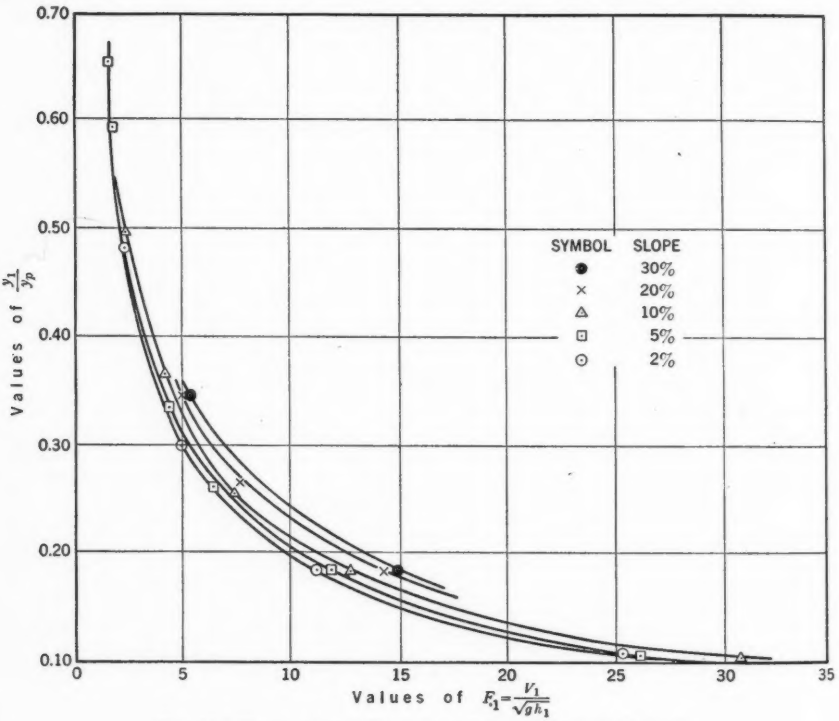
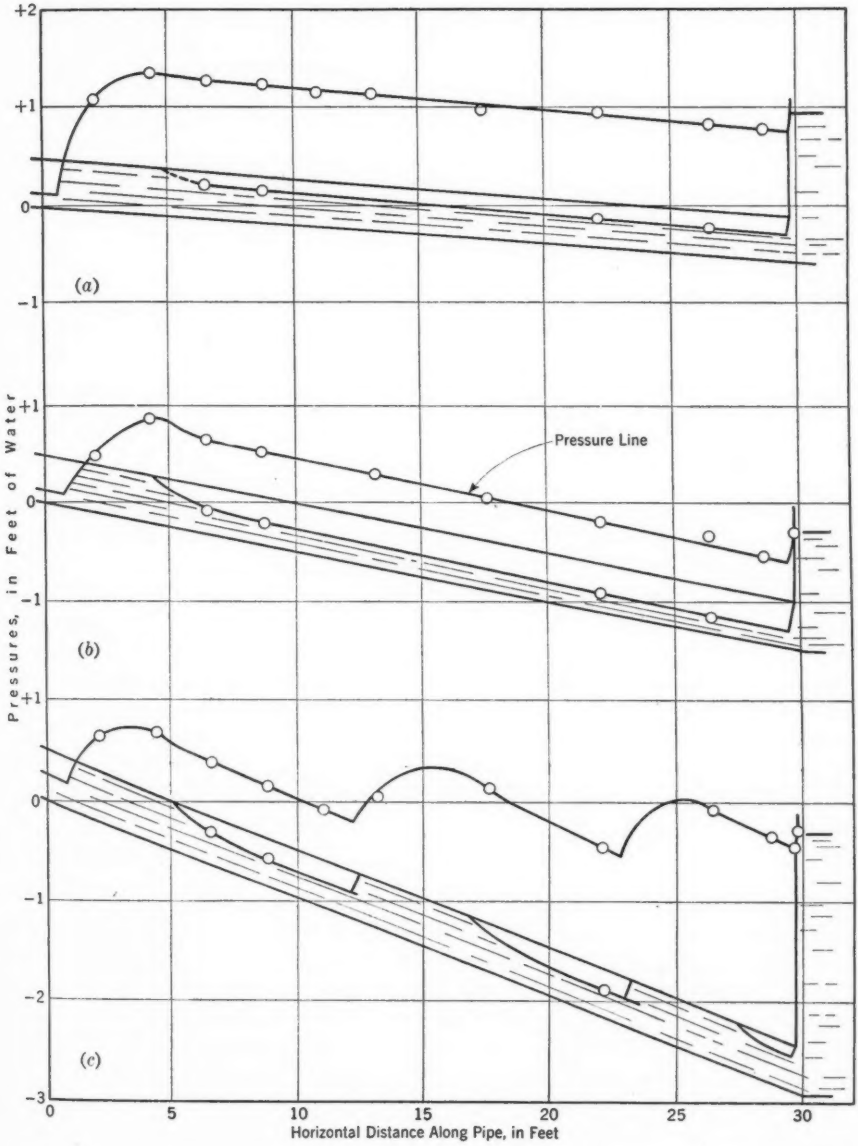



FIG. 14.—EXPERIMENTAL VALUES OF CRITICAL FROUDE NUMBER

In all the runs, water depth and pressure readings were taken along the pipe line (see Fig. 15). In most of the runs at the higher Froude numbers, the conduit beyond the jump had one long pocket of air under pressure. The water depth in this part corresponded to the normal depth of flow for that particular water discharge and slope. Since the velocity of the air flow in this pocket was relatively low the frictional losses due to the flowing air were negligible. In other words, the air in the pocket was under a constant pressure and thus the pressure line was parallel to the water surface in the pipe line which gradually became parallel to the pipe line. This is well shown by the data in Fig. 15.

In some of the runs made at the steeper slopes several consecutive stable jumps were obtained. Obviously if they remained stable each jump had to pump the same amount of air. The pressure variation with several jumps in




| Fig. | $y_1$ | $F_1$ | $Q_w$ | $Q_a$ | $S$ |
|---|---|---|---|---|---|
| (*a*) | 0.147 | 8.2 | 0.720 | 0.060 | 2% |
| (*b*) | 0.127 | 7.5 | 0.482 | 0.039 | 5% |
| (*c*) | 0.290 | 2.6 | 0.848 | 0.014 | 10% |

FIG. 15.—TYPICAL DATA ON WATER DEPTHS AND PRESSURES BEYOND HYDRAULIC JUMP IN 6-IN. PIPE LINE

the line is indicated by one of the pressure plottings in Fig. 15. Note how each jump increases the air pressure of each succeeding pocket; nevertheless, the pressure gradient still has a downward trend, as it must. The criterion which determines whether two or more stable jumps can form in a series depends on whether the normal depth of flow, $y_n$, for any discharge and pipe slope is less than $y_1$. As the ratio of $y_n$ to $y_1$ decreases, the closer together will be the jumps. Since each jump must pump the same quantity of air, the Froude number of the flow approaching each jump must be the same, and since $Q_w$ is constant, this means that the depth of flow ahead of each jump must be equal to $y_1$, the depth at the first jump. If one jump has a lower initial depth than the jump next below it, the first will send more air into the intervening air pocket than the second will take out. The two jumps will then move apart until the $y_1$ for the second jump is the same as that of the first. If the second jump has the smaller $y_1$ the action will be reversed and the jumps will move closer together.

## Verification of Momentum Principle

The jump in a closed conduit such as the one under consideration here can be analyzed by the use of the momentum principle in much the same manner as is done with the jump in open conduits. Knowing the initial velocity and depth, the pressure in the pipe when the conduit flows full just beyond the jump can be obtained. The use of the momentum principle for the hydraulic jump in horizontal closed circular conduits was verified[21] by E. W. Lane, M. Am. Soc. C. E., and C. E. Kindsvater, Jun. Am. Soc. C. E., in 1938. How the slope must be considered when applying the momentum relationship to hydraulic jumps in open channels has been indicated[22] by B. A. Bakhmeteff, M. Am. Soc. C. E., and A. E. Matzke, Assoc. M. Am. Soc. C. E.

Referring to Fig. 12, the general relationship for the present problem is as follows:

$$F_1 + M_1 = F_2 + M_2 - W \sin \theta \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

in which, $F_1$ and $F_2$ are the total pressure in pounds at sections (1) and (2), respectively; $M_1$ and $M_2$ are the total momenta of the water at sections (1) and (2), respectively; $W$ is the weight of the jump, and $\theta$ is the angle the conduit makes with the horizontal. The frictional effects at the conduit walls are neglected. The weight of the jump was computed from observations of the length of the jump, the area of the pipe $A_p$, and the unit weight of the water corrected for entrained air. The method of calculating the various terms in Eq. 35 was as follows:

(1) $F_1$ is the product of the water area $A_1$ at section (1) and the average unit pressure on this area which for a sloping pipe is $\gamma \bar{d}_1 \cos \theta$, in which $\bar{d}_1$ is the distance, perpendicular to the conduit, to the centroid of $A_1$, and $\gamma$ is the unit weight of water.

[21] "Hydraulic Jump in Enclosed Conduits," by E. W. Lane and C. E. Kindsvater, *Engineering News-Record*, Vol. 121, 1938, pp. 815–817.

[22] "The Hydraulic Jump in Sloped Channels," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, A. S. M. E., Vol. 60, 1938, pp. 111–118.

(2) $M_1$ is equal to $\frac{Q_w V_1 \gamma}{g}$.

(3) $F_2$ is computed from the measured value of the maximum pressure rise below the jump, $p_2$. The product of the pipe area and the average unit pressure on this area gives $F_2$, thus:

$$F_2 = \left( p_2 - \frac{\gamma D}{2} \cos \theta \right) A_p \ldots\ldots\ldots\ldots\ldots\ldots (36)$$

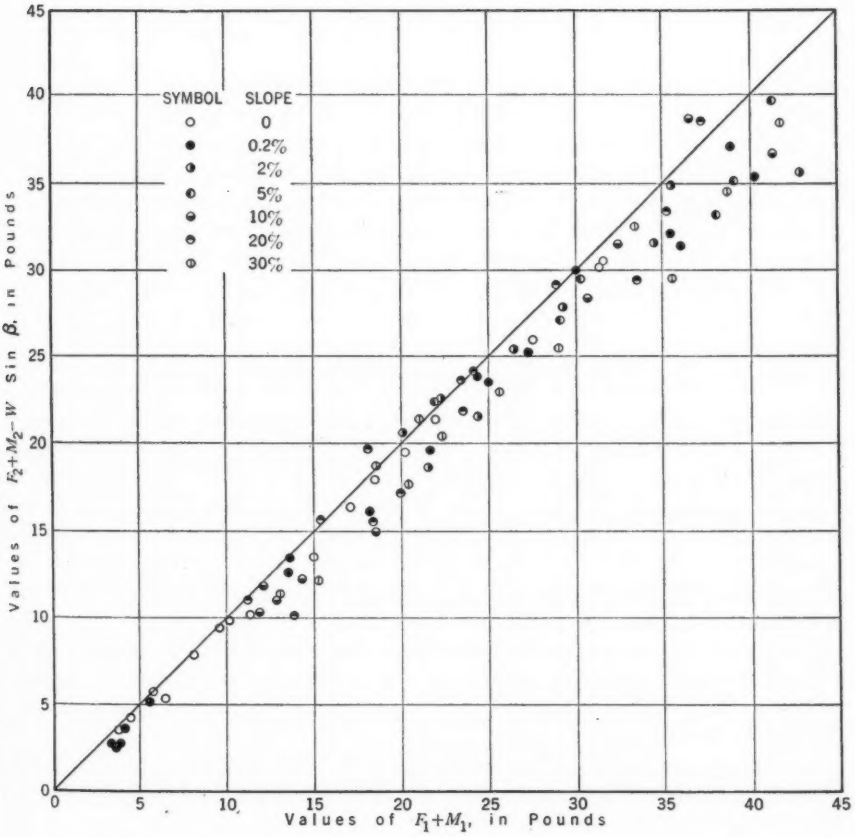



FIG. 16.—DATA CHECKING MOMENTUM RELATIONSHIP FOR HYDRAULIC JUMP IN 6-IN. PIPE

(4) $M_2$ is equal to $\frac{Q_w V_2 \gamma}{g}$. In calculating $V_2$, the fact that at section (2) the entire conduit is not filled with water should be taken into account; this can best be done by letting the area at section (2) be equal to $\frac{A_p}{1+\beta}$, in which

$\beta$ is the air-water flow (or volume) ratio. Therefore,

$$M_2 = \frac{\gamma\, Q^2_w\, (1 + \beta)}{g\, A_p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

(5) $W$ is equal to the product of the average length of the jump $L'$, the specific weight of the air and water mixture which can be expressed as $\frac{\gamma}{1 + \beta}$ and the pipe area:

$$W = A_p\, L' \left( \frac{\gamma}{1 + \beta} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

The value of $L'$ was taken as being approximately equal to the total observed length of the jump, $L$, minus $\frac{1}{2}(D - y_1)$.

In Fig. 16 the two sides of Eq. 35 are plotted, and the fact that the points group around a 45° line indicates that the data conform to this expression. The scatter of the data is due to difficulty in measuring values like depth of flow and pressure at the high velocities with any great degree of refinement. In any case the use of Eq. 35 is verified and especially the use of the term $W \sin \theta$, since for the steep slopes this term was as much as 75% of $F_1 + M_1$.

## Conclusions

The results of the experiments indicate that above a certain critical condition the rate of air removal from an air pocket in a pipe line will depend on the ability of the hydraulic jump which is formed to entrain air. This critical condition, for any slope of pipe and for any relative depth of flow in the air pocket, depends on the value of the Froude number of the flow ahead of the jump. Below this critical value of the Froude number the flow beyond the jump will not be able to handle all the air entrained by the jump, and thus the air removal will not be a function of the jump characteristics but rather on the hydraulic features of the flow beyond the jump.

Air flow measurements indicated that the rate of air entrainment by a hydraulic jump depended largely on the water discharge and the Froude number of the flow ahead of the jump. An empirical relationship is presented for the air flow.

Data on the hydraulic characteristics of the hydraulic jump in closed sloping circular conduits are presented. That the data can be correlated by the momentum relationship for the jump is verified.

## Acknowledgments

This paper largely summarizes a dissertation entitled "Entrainment of Air in Pipes by Flowing Water," submitted in August, 1941, by the junior writer to the Graduate College, State University of Iowa, Iowa City, Iowa, in partial fulfilment of the degree of Doctor of Philosophy in Hydraulics. The investigation was conducted under the direction of the senior author, as a project of

the Society's J. Waldo Smith Fellowship Fund. The complete original manuscript has been placed on file for reference at Engineering Societies Library.[22a]

The authors gratefully acknowledge the advice and suggestions of F. M. Dawson, M. Am. Soc. C. E., Dean, College of Engineering, State University of Iowa, and Mr. Kennison, Chairman of the J. Waldo Smith Fellowship Committee, and the other members of this Committee.

## APPENDIX

### NOTATION

The following letter symbols, adopted for standard use in this Symposium and subsequent discussion, conform essentially to American Standard Letter Symbols for Hydraulics[23] prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942.

$A$ = area:
- $A_a$ = cross section defined by a wetted perimeter due to water and entrained air;
- $A_c$ = cross section defined by a wetted perimeter due to water alone; $A_c = \frac{Q}{V}$;
- $A_o$ = observed cross-section area;
- $A_p$ = cross-section area defined by the inner surface of a closed conduit;
- $A_1$ cross section at sections (1), (2), (3), etc., are designated $A_1$, $A_2$, $A_3$, respectively;

$a$ = elemental area of a cross section;

$B$ = a substitution constant = $\frac{2q}{b}$;

$b$ = width:
- $b_c$ = water-surface width of a trapezoidal section;
- $b_1$ = bottom width of a trapezoidal section;

$C$ = Chézy coefficient = $\sqrt{\frac{\omega}{k}}$;

$D$ = inside diameter of conduit;

$d$ = distance: $\bar{d}_1$ = distance perpendicular to a conduit, to the centroid of the area $A_1$ at section (1);

$\mathbf{F}$ = Froude's number, subscripts referring to given sections; thus $\mathbf{F}_1$ = the Froude number of the flow approaching section (1);

$F$ = force; total pressure at a section denoted by appropriate subscripts;

[22a] 29 West 39th Street, New York, N. Y.

[23] ASA—Z10.2—1942.

$f$ = frictional resistance of the channel walls for a cross-section area of elemental length; subscripts 1, 2, 3, etc., refer to the corresponding sections;

$g$ = acceleration due to gravity;

$h$ = head:

$h_o$ = observed head;

$h_v$ = velocity head, computed from the mean velocity;

$h_z$ = potential energy head;

$K$ = coefficient of air entrainment: $K_1$ = a constant for air entrainment;

$k$ = coefficient of frictional resistance;

$L$ = total observed length of jump: $L'$ = average length of the jump;

$l$ = (see Fig. 9) length measured along channel bottom or closed conduit;

$M$ = total momenta of the water, subscripts indicating the section;

$m$ = parameter of parabola for trajectory of jet = $\frac{2\,V^2 \cos^2 \theta}{g}$;

$n$ = Kutter's or Manning's roughness coefficient:

$n_c$ = computed roughness coefficient to be used with $R_c$ and $V_o$;

$n_o$ = computed roughness coefficient to be used with $R_o$ and $V_c$;

$P$ = wetted perimeter for a mixture of air and water:

$P_c$ = computed wetted perimeter of a section of water only;

$P_o$ = observed wetted perimeter of a section of air and water;

$p$ = unit pressure (see $p_2$ in Eq. 36);

$Q$ = flow:

$Q_a$ = air discharge;

$Q_w$ = water discharge;

$q$ = unit discharge; flow per foot of width;

$R$ = hydraulic radius = $A/P$:

$R_c$ = computed hydraulic radius, $\frac{Q}{V\,P_c}$;

$R_o$ = observed hydraulic radius;

$r$ = radius of the center line of a channel;

$S$ = sine of angle between energy gradient and horizontal:

$S_c$ = sine of angle between energy gradient and horizontal for $h_v$ computed from $V_c$;

$S_o$ = sine of angle between energy gradient and horizontal for $h_v$ computed from $V_o$;

$S_1$ = sine of the slope of the channel bed;

$s$ = side slopes of a trapezoidal section;

$V$ = velocity, average in a section:

$V_a$ = average observed velocity in a section;

$V_c$ = computed velocity, $\frac{Q}{A_o}$;

$V_o$ = observed velocity;

$V_1$ = approach velocity, or velocity of water entering section (1) (subscript denoting appropriate section);

$v$ = velocity through an element of area;

$W$ = weight of the jump;

$x$ = horizontal lengths referred to an $xy$ plane: $x_t$ = horizontal component of trajectory of jet;

$y$ = depths, numerical subscript denoting section; vertical distances referred to an $xy$ plane:

- $y_a$ = depth of a stream comprising a mixture of air and water;
- $y_c$ = computed depth;
- $y_e$ = effective depth;
- $y_n$ = normal depth of flow;
- $y_o$ = observed depth;
- $y_t$ = vertical component of trajectory of jet;
- $y_1$ = initial depth = depth at section (1);

$\alpha$ = mean velocity-head coefficient used to obtain the average kinetic energy in a section;

$\beta$ = air-water flow ratio;

$\gamma$ = unit weight of water ($\omega$ is used by Mr. Hall);

$\theta$ = angle of channel bottom or pipe with horizontal, in degrees;

$\rho$ = ratio of water in a mixture of air and water or the relative density of the mixture; $1 - \rho$ = ratio of air in a mixture of air and water; and

$\psi$ = transverse slope of the water surface between the inner and outer walls of a channel.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

# FUNDAMENTAL ASPECTS OF THE DEPRECIATION PROBLEM

## A SYMPOSIUM

### Discussion

BY ROBERT F. LEGGET, ASSOC. M. AM. SOC. C. E., AND ERWIN E. HART, ESQ.

ROBERT F. LEGGET,[51] ASSOC. M. AM. SOC. C. E., AND ERWIN E. HART,[52] ESQ.[52a]—In the practice of engineering, depreciation is not the only subject to be much misunderstood due to the lack of a clear definition of meaning. It is among the most important of such matters, however, and one that inevitably will demand an increasing amount of attention. The appearance of this Symposium is therefore to be welcomed, especially since it is intended—in the words of Mr. Scharff—"to * * * make a new beginning toward formulating a preliminary agreement on fundamentals."

The three papers which constitute the Symposium make somewhat unequal contributions toward the objective set by Mr. Scharff. Although the writers agree with many of the individual statements of Mr. Walls, the general effect of his paper is but to confirm the complexity of the subject and the divergent views that are possible upon so many aspects of it. It will be generally agreed that "depreciation may well be a subject for engineering investigation and careful exercise of judgment." The fact that the Symposium is presented by the Society is tribute to this view; but it must also be admitted that factors other than those which can be classed strictly as engineering matters affect depreciation to a marked degree. Therefore, although engineers have a definite contribution to make to what may be called "the depreciation problem," theirs is not the only contribution, as Mr. Walls' paper would seem to infer.

NOTE.—This Symposium was published in November, 1941, *Proceedings*. Discussion on this Symposium has appeared in *Proceedings*, as follows: November, 1941, by Messrs. Edwin F. Wendt and L. T. Fleming, and Anson Marston; December, 1941, by Messrs. William G. Atwood, George E. Goldthwaite, Nathan B. Jacobs, H. J. Flagg, Nelson Lee Smith, and C. Beverley Benson; February, 1942, by Messrs. Paul T. Norton, Jr., H. L. Ripley, and Carroll A. Farwell; March, 1942, by Messrs. Thomas R. Agg, Conde B. McCullough, W. V. Burnell, Roger L. Morrison, Wallace B. Carr, W. L. Waters, A. G. Mott, David A. Kosh, E. G. Walker, and K. Lee Hyder; April, 1942, by Messrs. James T. Ryan, John C. Page, and John S. Worley; and June, 1942, by Messrs. John F. Miller, and Fred Asa Barnes.

[51] Asst. Prof., Dept. of Civ. Eng., The Univ. of Toronto, Toronto, Ont., Canada.

[52] Toronto, Ont., Canada.

[52a] Received by the Secretary June 4, 1942.

The view expressed by Mr. Walls that "depreciation * * * is an accomplished fact" is fundamental. It contrasts somewhat strangely with the suggestion under item (*d*) that the capacity of enterprises to make provision for depreciation will vary. Since depreciation is an accomplished fact at any one time, the provision for it is not, therefore, connected with the profitability of the enterprise as a whole. Provision for the latter is, rather, the contingency. Correspondingly, since depreciation is a fact, then "statistics" (using this term generally, as does Mr. Walls) will serve as an indication of improvement in the art currently taking place, and will be a guide at least to estimates of future life.

The writers are in complete agreement with the views expressed by Messrs. Crum and Winfrey. Their only regret is that, due to the lack of interest on the part of the average citizen in all governmental financial matters other than "reducing his taxes," they see little hope of the wide adoption in the near future of the sound principles advocated in this third paper. The appearance of the paper, and the indications it gives of the study now being given to this important aspect of public finance, give much encouragement to all who have at heart the interests of good government. One factor only does not appear to receive full consideration in this paper—namely, the fact that there are now, and there will be in increasing numbers, public works for which the benefits cannot be expressed wholly in financial terms. For such cases, some modification of general depreciation practice will have to be evolved.

Professor Grant's paper is interesting, provocative, and challenging. His examples are well chosen to illustrate the points he makes and the definitions he suggests. If the writers question certain features of his paper, as they must, they wish to preface such questionings with this genuine tribute.

Professor Grant starts his paper with three explanations of terms in common use in relation to depreciation—"popular depreciation," or decrease in value; "accounting depreciation," or amortized cost; and "appraisal depreciation," or the difference in value between the existing old asset and a hypothetical new asset. He proceeds to illustrate the use of these terms in the interesting cases that he cites in his paper. Example 1 is a particularly good illustration of the use of the three terms; it is referred to in the following note.

The writers wish to suggest that applying the word depreciation to these three values is somewhat confusing and represents one of the difficulties that must be cleared up before the matter can be brought to a proper basis. The values cited for popular depreciation and appraisal depreciation differ fundamentally from the value given as that determined by accounting depreciation, in that both of them are calculated at the actual time that the purchaser is considering buying the house. There is, therefore, little uncertainty about them except in so far as the appraisal depreciation makes a deduction for shorter life expectancy, etc. On the other hand, the value given for accounting depreciation was determined on the basis of calculations made in the year 1932 when no one could tell with certainty what the conditions with respect to property would be in the year 1940.

The writers believe that to compare these three values on an equal footing is scarcely a fair procedure. They suggest that in all discussions of depreciation

tion a clear distinction should be made between actual depreciations that can be calculated with respect to past time and estimates of depreciation that have to be made with respect to future time.

It seems to the writers that a fundamental understanding of depreciation is obtained by looking at the trend of annual or unit costs which the most efficient and modern machine designed for use in some particular field provides, using the lowest attractive interest rate that the class of investment involved may direct. In general, two things will be observed for most products:

(1) The price obtainable for the product will decline continuously—that is, the amount available from the sale of the product or service will decline from year to year, and thus the sum of: Operations ($O$) + maintenance ($M$) + depreciation ($D$) + interest ($I$) = total ($T$) must decline. This means that it is the sum of the last three ($M + D + I$) which must decline, as the first item may be taken to be constant.

(2) The trend of maintenance costs will be upward as the asset becomes older.

Thus there are two limiting conditions:

($a$) Due to the declining allowable total $T$, there will come a time when the allowable $T$ will equal the total of $O + M$, regardless of how slowly $M$ may rise. At this point the asset will cease to have value as an operating unit because its operating costs are relatively too high, and it thus has become "obsolete."

($b$) Due to the gradually rising $M$, there will come a time when the $O + M$ total alone will equal $T$, regardless of how slowly $T$ may fall. This may be defined as a statement of the extent of dilapidation.

Now any formula or arrangement for estimating either value or depreciation allowances must satisfy the series of figures associated with the two trends (1) and (2) and the two limits associated with ($a$) and ($b$). This yields a basic rule that the total of the charges $O + M + D + I$ must in general be steady, or decline from year to year, and must never rise.

Now apply this law to Professor Grant's formula under "Engineering Economy Studies Relative to a Proposed New Investment." Although, as Professor Grant indicates, it may be possible to arrange a series of annual payments to a depreciation account, which would end in establishing the same capital value as is given by this formula (Eq. 1), and not involve uniform annual payments, such a series would be extremely limited as to form and not at all likely to fit very many cases found in practice. Eq. 1 is thus essentially bound up with the idea of equal annual payments. The aforementioned rule thus limits its application to those cases where, for practical purposes, the curve of $T$ is a straight, horizontal line.

If Eq. 1 is used with an asset having other than a horizontal, straight-line relationship for $T$, then the totals of $T$ must be averaged, in which case the result will be only approximate. If the curve of $T$ is not horizontal and also not a straight line, the averaged result may be considerably in error. The latter condition is frequently found. In such a calculation it thus will be

essential to determine the curve of $M$ and take an average value of that too. This average will be open to the same risks as that of $T$ itself. Professor Grant has apparently taken $M$ into his value for annual disbursements in Examples 3 and 4. This is an equal and presumably averaged figure. In general, this formula would seem to be applicable to those assets having a horizontal, straight-line relationship for $T$, and at the same time having $M$ equal to zero, or forming a horizontal straight line. It is thus not generally applicable as an accurate formula.

TABLE 2.—EFFECT OF IRREGULAR MAINTENANCE TREND AND DECREASING ALLOWABLE COSTS (DOLLARS) ON THE ACCURACY OF EQ. 1

| Cost at: | | | Depreciation | Maintenance | Interest | Total |
|---|---|---|---|---|---|---|
| Beginning of year | End of year | Average | | | | |
| 2,000 | 1,600 | 1,800 | 400 | 0 | 108 | 508 |
| 1,600 | 1,300 | 1,450 | 300 | 75 | 87 | 462 |
| 1,300 | 1,100 | 1,250 | 200 | 150 | 75 | 425 |
| 1,100 | 950 | 1,025 | 150 | 175 | 62 | 387 |
| 950 | 800 | 875 | 150 | 175 | 53 | 378 |
| 800 | 650 | 725 | 150 | 175 | 44 | 369 |
| 650 | 525 | 588 | 125 | 200 | 35 | 360 |
| 525 | 400 | 463 | 125 | 200 | 28 | 353 |
| 400 | 275 | 338 | 125 | 200 | 20 | 340 |
| 275 | 175 | 225 | 100 | 225 | 14 | 339 |
| 175 | 75 | 125 | 100 | 225 | 8 | 332 |
| 75 | 0 | 38 | 75 | 250 | 2 | 327 |
| Total | .... | .... | 2,000 | 2,050 | .... | 4,580 |
| Annual | .... | .... | .... | 171 | .... | 382 |
| Maintenance deducted ($171) | | | | | | 211 |
| Sum added for interest to the beginning of the year | | | | | | 5 |
| Net cost as of the beginning of the year | | | | | | 216 |

Table 2 is an example of a case in which a decreasing series for $T$ is required (operating costs, being constant, have been omitted), and in which an irregular trend of maintenance costs is encountered.[53] It will be noted that this series will support a valuation of $2,000. For $n = 12$ and $i = 6\%$ in Eq. 1, the capital recovery factor is 8.576. The average annual net total of $216 if interest is taken on the investment at the beginning of the year, or $211 if interest is taken on the average investment during the year, will support a valuation of ($C = 216 \times 8.576 =$) $1,850 in the former case or ($C = 211 \times 8.576 =$) $1,810 in the latter. These values are 7.5% and 9.5%, respectively, below the actual worth of the asset. The formula assumes the former case. It thus would seem possible to obtain errors of as much as 10% or more with this formula. It will be noted that this is an error inherent in the formula, and consequently additional to any errors in the basic data, upon which the calculations were made. The writers realize that Eq. 1 provides a most valuable means of determining the value of many items of property. The object of this criticism is to indicate that errors are possible, nevertheless.

[53] "Depreciation and Its Influence on Unit Costs," by E. E. Hart, undergraduate thesis (unpublished) presented to the Faculty of Applied Science and Engineering, Univ. of Toronto, Toronto, Ont., Canada, 1939.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

# TIMBER FRICTION PILE FOUNDATIONS

## Discussion

BY A. E. CUMMINGS, M. AM. SOC. C. E.

A. E. CUMMINGS,[15] M. AM. SOC. C. E.[15a]—The question of the relative carrying capacities of single piles and pile groups has been a controversial subject among foundation engineers for many years. The experiments described in this paper might have served to add materially to the available knowledge on this problem. Unfortunately, it appears as though too much effort was devoted to the development of an unsound mathematical theory and too little effort to the development of a carefully thought out test program that would include field measurements of all of the quantities that are important in problems of this kind.

Referring to the mathematical theory, the author's effort to use the Boussinesq point-load equations simply adds another chapter to the long story of the incorrect application of these equations to all kinds of foundation problems. Several years ago, the writer published[16] the original derivation of the Boussinesq equations together with a statement of the fundamental assumptions on which the derivation is based. Whenever these equations are applied to an actual problem, it is necessary to keep these basic assumptions in mind and to make sure that the conditions of the problem do not conflict with the assumptions under which the equations were derived.

For example, one of Boussinesq's fundamental assumptions is that the load is applied at the plane boundary of the semi-infinite, elastic, isotropic, solid and not at some point below the surface of the solid. The author has used the Boussinesq equations for loads that are applied below the surface of the ground although it has already been explained by Raymond D. Mindlin,[17] Assoc. M. Am. Soc. C. E., why the Boussinesq equations cannot be used in this manner

NOTE.—This paper by Frank M. Masters, M. Am. Soc. C. E., was published in November, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by E. H. Connor, M. Am. Soc. C. E.; April, 1942, by Messrs. Glenn B. Woodruff, Jacob Feld, G. G. Greulich, and G. S. Paxson; May, 1942, by Raymond D. Mindlin, Assoc. M. Am. Soc. C. E.; and June, 1942, by John W. Solomon, Esq.

[15] Dist. Mgr., Raymond Concrete Pile Co., Chicago, Ill.

[15a] Received by the Secretary August 5, 1942.

[16] *Transactions*, Am. Soc. C. E., Vol. 103 (1938), p. 324.

[17] "On the Distribution of Stress Around a Pile," by R. D. Mindlin, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Cambridge, Mass., Vol. III, 1936, p. 71.

to determine the distribution of stresses around a pile. Furthermore, the author does not appear to have given any consideration to some of the other fundamental assumptions of the mathematical theory of elasticity on which the Boussinesq equations are based. In particular, the paper says nothing about strains or displacements, although these are just as important as the stresses in problems of this kind. Also, the author's extension of his analysis from the single pile to the pile group presupposes the validity of the principle of superposition. It is of considerable interest to examine the author's proposed theoretical solution from the standpoint of displacements and the principle of superposition.

As is well known, the entire mathematical theory of elasticity is based on Hooke's law concerning the proportionality of stress and strain. Whenever a force is applied to an elastic body, some kind of displacement occurs as a result of that force. The problem under consideration is no exception to this rule and the stresses determined by the author would necessarily be accompanied by displacements in the soil surrounding the pile. Since the author has assumed the soil to be incompressible (Poisson's ratio = 0.5), the horizontal tangential stress vanishes everywhere and the two normal stresses given by Eqs. 1*a* and 1*b* will determine the horizontal radial displacements by means of the equation

$$u = -\frac{r}{2E}(q_v + q_h) \qquad (20)$$

in which $u$ is the radial displacement, and $E$ is Young's modulus of the soil. The vertical displacements, $w$, may be obtained in a similar manner from the author's stress equations.

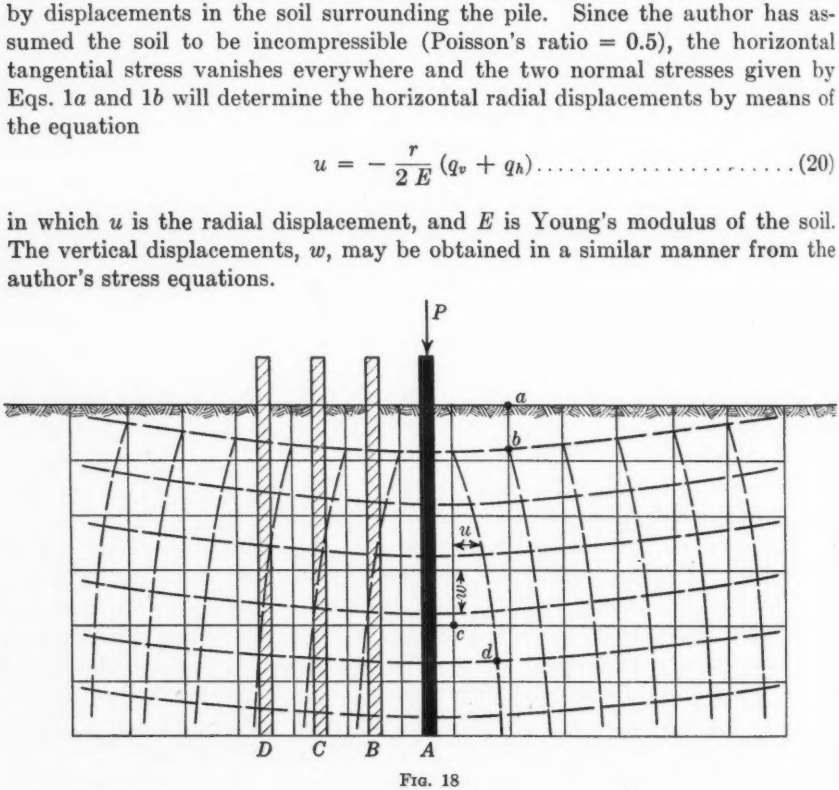


FIG. 18

Fig. 18 represents any meridional plane in the soil through the vertical axis of the pile A. The solid lines represent an imaginary rectangular network existing in the ground after the pile is driven and before any load is applied to

the pile head. When the load, $P$, is placed on pile A, stresses are set up in the soil around the pile, and these stresses are accompanied necessarily by certain vertical and radial displacements in the soil. The dotted lines in Fig. 18 are a qualitative representation of the distortions that would occur in the rectangular network due to the displacements determined by the author's theoretical stresses. The actual amounts of the displacements would depend on the magnitude of the load, $P$, and on the numerical value of Young's modulus, $E$, of the soil. All points on the ground surface would be displaced vertically downward and there would be no radial displacements at the ground surface because of the assumption of incompressibility (Poisson's ratio = 0.5). For example, a soil particle originally at a would move to b. Within the soil, all vertical cylindrical surfaces concentric with the vertical axis of the pile would be bulged radially outward and all horizontal planes would be bent downward. A soil particle originally at point c would move to point d.

This system of displacements is a necessary consequence of the author's stress system. If the stresses in the soil were those determined by the author, the displacements would have to be of the type shown in Fig. 18. It is conceivable that such a system of stresses and displacements could exist in the soil in the vicinity of a single pile. However, if the soil around the pile were subjected to restraints that would affect the displacement pattern, these restraints would also affect the stress pattern. This follows inevitably from the fact that the entire mathematical theory of elasticity is based on a definite set of stress-strain relationships. Boussinesq's stress equations were derived with the aid of the basic differential equations which relate the stresses to the space derivatives of the displacements. Accordingly, the Boussinesq stress pattern can exist only when it is accompanied by a certain definite pattern of displacements.

For the case illustrated by Fig. 18, the question arises as to what will happen to the stresses and displacements around pile A when that pile is surrounded by a group of other piles such as B, C, and D. The elastic properties of the piles will certainly be quite different from those of the surrounding soil regardless of whether the piles are made of wood, concrete, or steel. In effect, the piles will act as vertical reinforcement in the soil. They will distort the displacement pattern which is determined by the author's stress equations and this in turn will change the stresses themselves. Accordingly, it is not to be expected that the stress distribution around a single pile can be extended to a pile group by simple superposition as is done by the author in his theoretical analysis. Furthermore, it should be noted that the combination of soil and piles within the pile group is by no means the isotropic material postulated by Boussinesq in his analysis. The presence of the piles produces a compound material consisting of hard spots (piles) embedded in a matrix of much softer material (soil).

The principle of superposition is sometimes used to determine the stress distribution in foundation problems. For example, a large mat foundation is often divided into small segments and the stress distribution for each segment is computed as if the segment were a point load.[18] The stresses produced by

[18] *Proceedings*, Am. Soc. C. E., May, 1933, p. 784.

each segment are then added by simple superposition to obtain the stress distribution for the entire mat. This application of the principle of superposition to a mat foundation is something quite different from the application of that principle to a pile group. In the case of the pile group, each additional pile affects the elastic properties of the underground. Even though the soil around a single pile might be considered as having the properties of an elastic isotropic material, the introduction of additional piles destroys the isotropic properties and materially changes the elastic properties. Accordingly, the author's method of applying the principle of superposition to a pile group is an incorrect procedure.

Another factor in this problem to which the author has paid no attention is the contraction of the pile itself. After the pile is in place in the ground and before any load has been applied to the pile head, there is no shearing stress in the soil such as that given by Eq. 2*c*. When a load is applied to the pile head, there is a downward movement of the upper end of the pile due to the elastic shortening of the pile. As the pile starts to move with respect to the surrounding soil, the movement mobilizes the shearing resistance of the soil. Since the movement begins at the top of the pile, the mobilization of shearing resistance in the soil also begins at the top of the pile near the ground surface. When a small load is applied to a long friction pile, all of the reaction for the load is developed in the soil around the upper part of the pile. The load produces no stress in the lower part of the pile and consequently no elastic contraction and no mobilization of shearing resistance in the soil surrounding the lower part of the pile. As the load is increased the stress in the pile approaches the pile point but it does not actually reach the point until the load attains a certain value which is determined by the elastic properties of the pile and the surrounding soil.

In Fig. 6, the author has computed the shearing stresses around a single pile 60 ft long carrying a 10-ton load. A value of $R_c = 1.5$ is used in the calculation. Actually the value of $R_c$ would be very much greater than 1.5 for the conditions shown in the figure because little or no load would be delivered to the soil at the bottom of the pile. If no load is delivered to the soil at the bottom of the pile, the value of $R_c$ becomes infinite. As a matter of fact, the determination of the numerical value of $R_c$ should be the starting point for any theory of friction pile behavior. The real problem is to determine how the pile delivers its load to the surrounding soil. The author states that $R_c$ is a function of pile length, pile taper, and internal friction of the soil. He has overlooked the fact that it is also a function of the elastic properties of the pile as well as a function of the load that is applied to the pile head. If the pile were perfectly rigid so that no elastic shortening of the pile would occur under any load, it might be possible to have $R_c = 1.5$ for all loads and to have a shear-stress distribution similar to that shown around the single pile in Fig. 6. Since all actual piles are elastic and will therefore shorten under compressive stresses, this elastic shortening will influence the transfer of load from the pile to the soil. The value of $R_c$, therefore, will depend on the elastic properties of the pile and on the amount of the applied load as well as on the three factors mentioned by the author.

The usual procedure in the case of a pile load test is to place the load on top of the pile and then to measure the downward movement of the head of the pile for each load increment. It is not possible to determine the elastic shortening of the pile from such measurements because nothing is known about the movement of the pile point. Neither can such measurements furnish any information as to the manner in which the pile transfers the load to the surrounding soil. On the other hand, if the actual elastic shortening of the pile were determined by suitable measurements, it would be possible to determine from these measurements approximately how the pile was transferring its load to the soil. For example, it is a simple matter to compute what the elastic shortening of the pile would be if it were a free-standing strut with no part of the load supported by friction on the sides of the strut. When any part of the load is supported by friction along the sides of the pile, the elastic shortening is always less than it would be for the same pile carrying the same load as a free-standing strut. When all of the load is carried by side friction as in the author's problem, it is possible to compute the elastic shortening of the pile for any assumed distribution of friction along the sides of the pile. This can be done even if the friction distribution is assumed to be discontinuous due to soil stratification. Conversely, if the elastic shortening of the pile is carefully measured in the field, it is possible to deduce the friction distribution from the field measurements.

Experiments of this kind have been made in The Netherlands by C. Franx,[19] G. C. Boonstra,[20] and others. The piles tested in The Netherlands were pre-cast concrete piles. When they were cast, a piece of 1.5-in. pipe extending the full length of the pile was placed in the vertical axis. A 1-in. rod was inserted in this pipe and fastened at the pile point. The test load was applied to the driven pile in such a manner that it was possible to measure the deflection of the pile head and also the movements of the 1-in. rod which gave the deflections of the pile point. The difference between the two readings for any given load represented the elastic shortening of the pile at that load. In Fig. 19, curves a and b are reproductions of a pair of load-settlement curves measured by Boonstra. The solid line shows the settlement of the pile head and the broken line that of the pile point. The specimen was a reinforced pre-cast concrete pile of octagonal cross section, 16 in. in diameter and 70 ft long. It was driven through 50 ft of soft clay and peat, 15 ft of fine sand, and then into sand and gravel for a depth of about 6 ft. The condition was similar to that of pile T20 of the author's tests.

As can be seen from Fig. 19, there was practically no movement of the pile point until the load amounted to 15 or 20 tons although the head of the pile began to move as soon as the loading was started. The vertical distances between the two curves, a and b, represent the elastic shortening of the pile, but in order to show this relationship more clearly, curve c, Fig. 19, was plotted with loads as abscissas and elastic shortening as ordinates. It is easily seen from this figure that the relationship between load and elastic shortening is not linear. If the distribution of load transfer from pile to soil were constant

[19] "Proefbelasting van Heipalen te Rotterdam," by C. Franx, *Polytechnisch Weekblad*, 1934, Nos. 23, 24, and 25.

[20] "Eenige Beschouwingen over den Puntweerstand van Palen," by G. C. Boonstra, *De Ingenieur*, February 23, 1940, Vol. 55, No. 8, p. B-33.

throughout the entire load range ($R_c$ = constant), and if the pile material were perfectly elastic, curve c, Fig. 19, would be a straight line. As to the elastic properties of the pile, it could be expected to behave under load like a reinforced concrete column. The stress-strain curve for such a column might be slightly curved but the curve would be concave toward the strain axis, whereas most of curve c, Fig. 19, is concave toward the stress axis. Accordingly, the only conclusion that can be drawn from curve c is that the distribution of load transfer from pile to soil varies throughout the load range. In other words, $R_c$ is not a constant. As the load is increased, the transfer of load from pile to soil distributes itself in various ways depending on the elastic properties of the pile and the soil and on the amount of the load. The sudden break in curve c at about 150 tons indicates that the upper part of the pile probably broke loose from the surrounding soil entirely so that practically all of the applied load was resisted by the lower end of the pile which was in the sand and gravel.

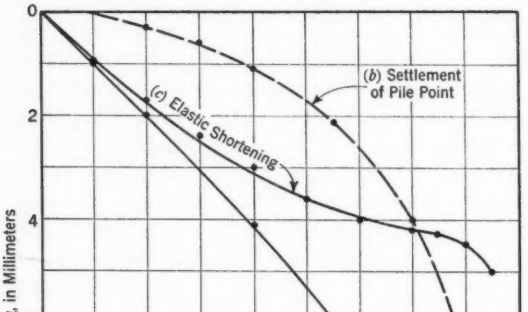


FIG. 19

Consideration of this type of test data raises a serious question as to the validity of the figures presented in Table 8. All of the pile loads given in this table are computed with $R_c = 1.5$ on the assumption that the distribution of load transfer from pile to soil is constant over a wide range of loads and pile lengths. The author's computation implies that $R_c$ is an independent variable to which numerical values can be assigned for the purpose of computing pile loads. Actually, the situation is just the reverse. The amount of the load influences the manner in which the pile transfers load to the soil so that the numerical value of $R_c$ depends on the load.

This aspect of the problem is something that could easily have been investigated in the Morganza tests. It would have been possible to set three or four rods in a hollow concrete pile, such as pile C9, so that settlement observations could have been recorded not only at the pile head but also at the pile

point and at several places along the length of the pile. If the strains in the pile itself had been measured, it would be possible to determine how the pile was transferring load to the soil at each load increment.

Because he has completely neglected the entire question of strains and displacements, the author has developed a mathematical analysis that is theoretically unsound and of no practical value. The analysis is based on the mathematical theory of elasticity but nowhere in the paper is there any mention of the elastic properties of the piles themselves. The author has treated the piles as though they did not exist except as imaginary vertical line loads in the soil. Unfortunately, this method has led to a theoretical stress distribution which, in the writer's opinion, must also be classified as imaginary. The piles are actual physical bodies having certain elastic properties, and neither the single pile nor the pile group can be analyzed without taking these elastic properties into account.

Throughout his theoretical analysis of pile groups the author has assumed that each pile in the group would be subjected to the same load as all of the others. He has made this same assumption in connection with the groups that were actually tested. This question of load distribution over the pile heads is an important factor in pile group behavior. The author has referred to this subject in his conclusions by remarking that the piles under the outer edges of a stiff pier base would be subjected to greater loads than those under the center. This remark is perfectly true but the question arises as to what was happening in this respect during the Morganza group tests.

Fig. 2(*c*) shows the load arrangement that was used for the 16-pile group. If each pile in the group were subjected to the same load, the center piles could be expected to settle the most, the edge piles would settle somewhat less, and the corner piles would settle least of all. The photograph indicates that the applied load was quite rigid so that it could scarcely be expected to warp in a manner that would conform with the settlements that would be produced by equal pile loads. Instead, the rigidity of the test apparatus would require equal settlement of the pile heads and this in turn would affect the distribution of load to the piles. A perfectly rigid structure would put the greatest load on the corner piles and the least load on the center piles.

This is another aspect of the problem that could have been investigated in the Morganza group tests. Fig. 2 indicates that the pile heads were several feet above the ground surface and it would have been possible to make strain-gage measurements at the upper ends of the piles to determine how much load each pile was carrying. There is no justification for the assumption that all of the piles in a group were carrying equal loads since no effort was made to produce this condition. To insure equal loads on all piles in a group it would be necessary to support the test load independently of the pile group and then to load all of the piles simultaneously with individual hydraulic jacks reacting against the under side of the load platform. It should be kept in mind that the distribution of pressure in the contact plane between any two solid bodies will depend on the relative elastic properties of the two bodies. This applies to loaded pile groups as well as to loaded masses of earth or other materials.

Furthermore, it should be noted that this phenomenon could explain the behavior of the Morganza pile groups without reference to the author's theory of stress distribution. The practically rigid structures built on top of the pile groups could have had the effect of overloading the corner and edge piles so that the groups would have failed by leaning toward a corner or side as described by the author. No theory of pile group behavior can be considered as being complete or even approximately correct unless this phase of the problem is taken into account.

To demonstrate the accuracy of his theory, the author has prepared Table 5, which indicates a remarkably close agreement between theoretical and actual failure loads. However, the writer finds it difficult to follow the line of reasoning used by the author in his selection of shear values on which the comparisons are based. An average value is used for four of the group tests, and the other three are compared with values determined for certain individual piles. Referring to Fig. 1, these two profiles represent cross sections of the floodway 4 or 5 miles wide. The profiles themselves are some miles apart along the length of the floodway. The single piles, as well as the pile groups, are therefore scattered over distances that are measured in miles, and the first question to be decided is whether the comparisons should be based on averages or whether each group test should be compared with the results of the nearest single pile.

It is of considerable interest to examine Table 5 with these ideas in mind. To check the T2–T9 group, the author used pile T1 which was 130 ft from the group. However, pile T10 was only 294 ft from the group. Accordingly, if the proper way to handle this problem is by means of averages, it would seem reasonable to average T1 and T10 in order to check this group. Such a procedure would not check the theory so closely because the average friction developed by T1 and T10 was 792 lb per sq ft, whereas the author used 685 lb per sq ft from T1 alone. On the other hand, if the proper way to handle this problem is by comparing each group with the nearest single pile, groups T11–T19, T21–T29 and T37–T52 should be compared with T20 alone and not with an average that includes piles that are miles away from these groups. The distance from group T11–T19 to group T21–T29 is 1,958 ft and the third group as well as pile T20 are between the two. However, a comparison of these groups with pile T20 would not check the theory so well because pile T20 developed a friction value of only 680 lb per sq ft, whereas the author checked these groups by using the general average value of 778 lb per sq ft.

The data given in Table 5 indicate that the tests provided an almost perfect check of the author's theory. The agreement between theoretical and actual loads is really too perfect in view of the fact that the theory is unsound from the point of view of both mathematics and theoretical mechanics. In the writer's opinion, the author's theory is analogous to the old Ptolemaic theory of a geocentric solar system. For 1,400 years the Ptolemaic system was accepted generally because it appeared to be able to account for the motions of the planets. After Galileo invented the astronomical telescope, astronomers were able to stop speculating and start measuring. The very first measurements caused the abandonment of Ptolemy's geocentric system in favor of the

modern heliocentric system. The fact that Galileo got into trouble for measuring things and upsetting an ancient theory should not discourage modern engineers from making all the measurements that can possibly be made in a given experiment. A theory based on insufficient data can appear to be practically perfect and yet be completely wrong.

The writer joins the author in the hope that engineers interested in pile foundations will continue to make tests to determine how such foundations behave under load. However, the writer considers it necessary to supplement that hope with several suggestions. To develop an adequate theory for the behavior of a single friction pile, it will be necessary to determine, by means of suitable field measurements, how the pile transfers its load to the surrounding soil. In order to develop an adequate theory for the behavior of a group of friction piles, field measurements must be made to determine the distribution of load to the pile heads in addition to the measurements that are required to determine the distribution of load transfer from pile to soil. Field measurements that include only a record of the settlement and rebound of the pile head are inadequate for the development of a theory of friction pile behavior.

Correction for *Transactions:* In June, 1942, *Proceedings*, Eq. 7*b*, change "$dL'$" to "$dZ$"; on page 1049, line 10, change "$P_1$" to "$p_1$"; in Table 16, change "0.035" and "0.095[a]" to "0.0035" and "0.0095[a]," respectively.

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DISCUSSIONS

# DESIGN OF ST. GEORGES TIED ARCH SPAN

## Discussion

BY MESSRS. A. A. EREMIN, AND A. M. FREUDENTHAL

A. A. EREMIN,[10] ASSOC. M. AM. SOC. C. E.[10a]—Analysis of stresses in the arch rib tied with a stiffening girder is an extremely complicated problem. In the technical literature very little can be found on this subject. In computing the stresses in the St. Georges arch bridge, Professor Garrelts has adopted various assumptions to simplify the computations. The writer proposes to clarify, further, the limitations of assumptions adopted in the light of economic aspects of tied arch bridges with stiffening girders.

From Fig. 6 it is evident that the stiffening girder in the St. Georges bridge is connected to the arch rib by a rigid joint, whereas, in the development of Eq. 3 for the computation of stresses in the arch rib, the author assumed that the stiffening girder is connected to the arch rib by a frictionless hinge. This assumption has a negligible effect on computation of the horizontal thrust in the arch rib, especially for ratios $\frac{h}{L}$ equal to or greater than $\frac{1}{10}$. However, the equations developed with this assumption are not adequate for the design of the arch rib sections at the ends and their connections to the stiffening girder.

The axial stresses in the stiffening girder, produced by the arch rib thrust, are generally small compared with the bending moment stresses. Therefore, sections of stiffening girder are usually varied according to variations of the bending moment stresses along the girder axis. In developing Eq. 17 for the horizontal thrust in the arch rib, Professor Garrelts, however, assumed that the stiffening girder section is constant for the entire length. Evidently with this approximation the refinement of considering that the arch rib section varies directly as $\cos \alpha$ (Eqs. 16) has doubtful meaning.

In the design of tied arch rib bridges, an important consideration is the design of the lateral stiffening system. Often economy in material gained by

NOTE.—This paper by J. M. Garrelts, Assoc. M. Am. Soc. C. E., was published in December, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by R. W. Abbett, M. Am. Soc. C. E.; April, 1942, by Jacob Karol, Esq.; May, 1942, by Alexander Dodge, Esq.; and June, 1942, by Carlos A. Bejarano, Jun. Am. Soc. C. E.

[10] Associate Bridge Engr., Bridge Dept., Div. of Highways, State Dept. of Public Works, Sacramento, Calif.

[10a] Received by the Secretary June 22, 1942.

careful computations of the stresses in the arch rib and stiffening girder may easily be lost in the wasteful approximate design of the lateral stiffening system. It is to be hoped that, in his closing comments, the author will discuss a method of designing the lateral stiffening system and will present some principles to govern the selection of panel lengths.

The author is to be commended for making a valuable contribution to this subject.

A. M. FREUDENTHAL,[11] ASSOC. M. AM. SOC. C. E.[11a]—When considering the formulas for the design of the tied arch derived by Professor Garrelts, the question immediately arises as to whether the validity of these formulas is not subject to the same limitations as the formulas of the elementary theory of the suspension bridge, particularly taking into consideration the relatively large span of the structure actually designed. High deformability is a common feature of all structures in which a shallow girder suspended from, or supported by, a flexible member provides the bending resistance for unevenly distributed live load. It is this deformability that renders the methods of the elastic theory of structures ineffective in the design of suspension bridges as well as of arches, particularly those with flexible ribs or of long spans.[12]

In general, designers are quite familiar with the fact that considerable differences are to be expected between the bending moments in the stiffening truss of a suspension bridge, computed by the elastic theory, and the actual moments. They are less familiar with the effects of the deflections upon the bending moments in arches. Therefore, a few quite general remarks regarding the limitations imposed upon the validity of elementary formulas by the flexibility of a structure may not be devoid of interest. In no respect do these comments deprecate the method of design presented by the author which, incidentally, is practically unaffected by considerations of deflections.

In non-tied flexible systems (suspension bridges and arches) the bending moment due to live load can be expressed by

$$M = M'_p - H_p\, y \mp (H_w + H_p)\, \eta \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

in which $p$ and $w$ denote statical values due to live load and dead load, respectively, and $\eta$ denotes the deflection of the cross section of the girder considered. Hence, the moments in the stiffening girder computed by the elastic theory are either reduced (suspension bridges) or increased (arches) by the amount $(H_w + H_p)\, \eta$. Owing to the considerable value of the dead-load thrust $H_w$ for large-span structures, the deviation from the elastic theory may attain a comparatively high proportion of the total bending moment.

It can be easily shown that in self-tied structures the bending moment in the girder (Fig. 14),

$$M = M'_w + M'_p - (H_w + H_p)\, (y \pm \eta) \mp (H_w + H_p)\, \eta = M'_p - H_p\, y \ldots (42)$$

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[11a] Received by the Secretary August 1, 1942.

[12] "Deflection Theory for Arches," by A. M. Freudenthal, Publications of the International Assn. of Bridge and Structural Engineering, Zürich, Vol. 3, 1935, p. 100.

is not affected by deflection if the flexible member (the arch rib or the cable) has no bending resistance. However, if the bending resistance cannot be neglected (as in most arches), the effect of the deflections upon the value of the bending moments makes itself felt in the rib moments. The real values of

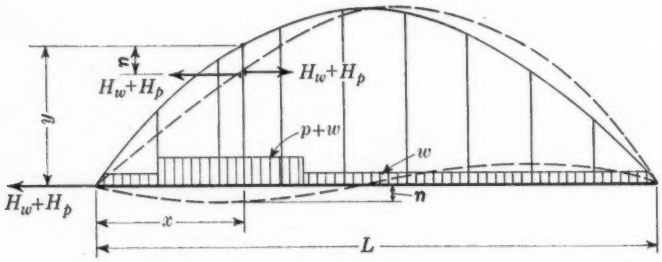


FIG. 14

these moments, therefore, may exceed, considerably, those obtained by simple distribution of the total bending moments between rib and girder according to the respective moments of inertia, as proposed by Professor Garrelts. It is particularly the buckling resistance of the rib that may be impaired by a bending moment exceeding the design value.

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DISCUSSIONS

# HYDRAULIC DESIGN OF DROP STRUCTURES FOR GULLY CONTROL

## Discussion

BY MESSRS. N. A. CHRISTENSEN AND DWIGHT GUNDER, AND BORIS A. BAKHMETEFF AND NICHOLAS V. FEODOROFF

N. A. CHRISTENSEN,[18] ASSOC. M. AM. SOC. C. E., AND DWIGHT GUNDER,[19] ESQ.[19a]—The exceedingly large number of drop structures needed in gully-control and irrigation work is reason enough for laboratory research of this kind. The actual value of the research becomes very evident when one considers the large number of failures of drops already built and also the large number which have been overbuilt in order to insure safety. The value of this research has already been demonstrated; but, if another wave of conservation and reclamation sweeps the United States after World War II, these researches will pay unusually large dividends.

The "half model" technique used in these experiments deserves further elaboration. The two principal advantages of this technique are:

(1) It permits accurate observation of conditions at the center line of the channel; and

(2) With a given model size, the quantity of water required is reduced by half.

The objections which might be raised are:

(1) Extraneous friction forces are introduced by the glass plate; and

(2) The turbulent action normally found at the center of the stream is eliminated by the rigid glass plate serving as a boundary.

The two advantages enumerated are obvious and need no further discussion. The following remarks will be confined to the disadvantages mentioned.

NOTE.—This paper by B. T. Morris, Jun. Am. Soc. C. E., and D. C. Johnson, Assoc. M. Am. Soc. C. E., was published in January, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1942, by John Hedberg, Assoc. M. Am. Soc. C. E.; May, 1942, by Messrs. L. Standish Hall, and J. E. Christiansen; and June, 1942, by Walter T. Wilson, Assoc. M. Am. Soc. C. E.

[18] Dean of Eng., and Chairman Eng. Div., Experiment Station, Colorado State College of Agriculture and Mechanic Arts, Fort Collins, Colo.

[19] Assoc. Prof. of Civ. Eng., Colorado State College of Agriculture and Mechanic Arts, Fort Collins, Colo.

[19a] Received by the Secretary June 29, 1942.

The friction forces introduced by the glass plate are a source of error, but in every model study of this kind friction forces are always a source of error because Froude's number is used as a criterion for similarity. With this number as a criterion, one is assured of the proportionality of the gravity and inertia forces in the prototype and model. The friction forces, on the other hand, are entirely neglected and considered insignificant as far as results are concerned. The smaller the model, the more serious is the error introduced by this assumption. It is quite likely that the increase in scale made possible by the "half model" technique more than compensates for the extra friction force introduced by the presence of the glass plate at the center line.

In order to appraise the importance of the second objection to the "half model" technique, attention is called to natural channels in which drop structures have been erected. From a study of these natural channels, it is obvious that the excessive scour which sometimes occurs just below the structure is caused by lateral currents of the stream itself and not by the haphazard turbulence which is present in most streams. If normal stream turbulence were the cause of the excessive scour, it would not be localized just downstream from the drop structure but would occur equally along the entire reach of the stream. These lateral currents and excessive scour are invariably associated with an unlevel water surface and the pressure gradients within the body of water which cause continuous currents having lateral velocity components. In most drop structures, a vertical plane of symmetry exists, which includes the center line of the structure. Not only is the structure itself symmetrical about this plane, but also the mean flow and the pressure field within the flow. In such structures a vertical plane may be inserted at the center line, and the mean flow will be essentially unchanged. It is true that the turbulent flow components will be reduced, but these components play only a secondary part in the excessive erosion that sometimes takes place just downstream from the drop structure.

The requirements necessary for the "half model" technique would seem to be:

(*a*) Geometric symmetry about the center line of the structure;
(*b*) Kinematic and dynamic symmetry of the mean flow; and
(*c*) Stability of the flow pattern normal to the plate at the center line.

Whenever these three conditions do not obtain, the "half model" technique should not be used.

In the paper, an inference was made that gully-control structures designed according to the principles set forth would require very little maintenance. For a series of drop structures to be permanently and completely successful, each structure must not only be soundly designed, but the reaches between the structures must also be completely stable. The question of what is a stable grade for the gully must first be answered before the height and spacing of the various drops can be determined. The importance of an accurate determination of a stable grade under the varying flow conditions of the gully should be emphasized as part of the over-all problem of gully control by means of structures. This problem is further complicated by the fact that the flow

conditions of gullies may change rather rapidly due to various erosion-control practices.

The writers would like to request elaboration of certain parts of the paper. The statement made beginning nine lines from Eq. 7*c*—"The height of the end sill of the stilling pool necessary to develop energy-dissipating action is apparently determined by the depth and energy content of the flow in the pool"—seems to be somewhat inconsistent with Eq. 7*c*. This formula would call for the same sill height regardless of the height of drop. The height of drop, on the other hand, has much to do with the energy content of the flow in the pool. Eq. 7*c* seems to indicate that the height of the sill is determined by $d_c$ alone, which in turn is a direct measure of quantity of flow. As far as Eq. 7*c* is concerned, the question of energy does not enter.

The last sentence of the third design rule (see heading "Summary of Design Rules and Formulas") seems to be somewhat inconsistent with the material contained in the article proper. The discussion involving Eqs. 3 to 6, as well as Fig. 14, seems to indicate that Eq. 6*a* would apply for values of $\frac{h}{d_c}$ smaller than 1. If the use of this formula is hazardous for values of $\frac{h}{d_c}$ smaller than 1, then a certain part of Fig. 14 should be deleted.

It should be noted also that, whereas the derivation of Eq. 6*a* starts with what appears to be a rational attack, the final equation is almost completely empirical, especially so in that it contains a purely empirical term. Further, the expression for the thickness of the falling sheet (given as proportional to $d_c\sqrt{\frac{d_c}{h}}$) is a rather rough approximation, which is satisfactory only if $\frac{d_c}{h}$ is relatively small. It can be seen at once that this expression, $d_c\sqrt{\frac{d_c}{h}}$, for the thickness approaches infinity as $h$ becomes smaller. One is thus inclined to suspect that any equation using this thickness will be inaccurate in the region where $\frac{d_c}{h}$ is large. This is just what was found by the authors.

A slightly different attack on this problem will now be given. If the depth $d_c$ is included in the calculation of the location of the point of impingement of the jet on the floor, and the other assumptions are as given in the paper, it is found that the horizontal distance $x$ from the crest to the middle of the impinging sheet is $x = \sqrt{d_c}\,(2\,h + d_c)$. The thickness of the falling sheet under the same conditions is given by the continuity equation as

$$t = d_c \frac{\sqrt{d_c}}{\sqrt{2\,(h + d_c)}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17a)$$

and the horizontal width of this sheet is given by

$$w = t \sec\theta = \frac{d_c\sqrt{d_c}}{\sqrt{2\,(h + d_c)}} \times \frac{\sqrt{2\,g\,(h + d_c)}}{\sqrt{2\,g\left(h + \frac{d_c}{2}\right)}} = \frac{d_c}{\sqrt{2\,\frac{h}{d_c} + 1}} \qquad \ldots (17b)$$

in which $\theta$ is the angle between the vertical and the direction of the sheet filaments.

If one now assumes that the length of the apron is linearly dependent upon each of these equations ($C$ and $C_1$ are empirical factors):

$$L = C\sqrt{d_c\,(2\,h + d_c)} + C_1 \frac{d_c}{\sqrt{2\,\frac{h}{d_c} + 1}}$$

$$= \left( C + \frac{C_1}{1 + 2\,\frac{h}{d_c}} \right) \sqrt{d_c\,(2\,h + d_c)} \quad \ldots\ldots\ldots\ldots\ldots (18)$$

Since the writers did not have the original data available for computations, a rough comparison of the apron lengths given by Eq. 18 and by Eq. 6*a* was made by assuming $C = C_1$. This gives

$$L = 2\,C\,\frac{\left(1 + \frac{h}{d_c}\right)\sqrt{d_c\,(2\,h + d_c)}}{1 + 2\,\frac{h}{d_c}} = 2\,C\,\frac{\left(1 + \frac{d_c}{h}\right)}{\sqrt{2 + \frac{d_c}{h}}}\sqrt{h\,d_c} = C'_L\sqrt{h\,d_c} \quad \ldots (19)$$

The coefficient $C'_L$ of $\sqrt{h\,d_c}$ in Eq. 19 corresponds to the $C_L$ of the author's paper. Using $2\,C = 3.5$, the values of $C'_L$ were computed and are compared with the values of $C_L$ estimated in Fig. 14 (see Table 4). These results show a maximum variation of $C'_L$ of about 5% from the values of $C_L$ (excluding the doubtful value $\frac{d_c}{h} = 1.2$). Since this is accomplished without the inclusion of a purely empirical term such as the $0.7\left(\frac{d_c}{h}\right)^3$ in Eq. 6*a*, it might be considered as somewhat better than Eq. 6*a*.

TABLE 4.—VALUES OF $C'_L$ COMPARED WITH VALUES OF $C_L$ ESTIMATED IN FIG. 14

| Coefficient | Ratio, $\frac{d_c}{h}$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2[a] |
| $C'_L$ | 2.5 | 2.8 | 3.15 | 3.45 | 3.8 | 4.1 | 4.35[a] |
| $C_L$ | 2.5 | 2.7 | 3.0 | 3.3 | 3.75 | 4.3 | 5.0[a] |

[a] These values are in the range where the authors claim that $C_L$ should not be used. The writers do not know what values should be used.

It should be noted that the foregoing results were obtained with just one empirical constant. Perhaps even better results could be obtained by adjusting both $C$ and $C_1$.

The writers do not wish to claim that this is a rigorous rational derivation of the expression for apron length $L$, but it is felt that it may be helpful in further study of the problem of the design of gully-control structures, which the authors have so admirably attacked.

BORIS A. BAKHMETEFF,[20] M. AM. SOC. C. E., AND NICHOLAS V. FEODOROFF,[21] ESQ.[21a]—The paper presents an interesting endeavor to proportion structural design by basing the dimensioning on data derived from detailed observations of the physical aspects of the flow. It is evident from Fig. 6 that the dimensions of the stilling pool are principally conditioned by the horizontal distance from the wall at which the falling live vein strikes the apron, and implicitly by the angle at which such striking is effected. The latter seems in particular to determine the direction followed by the rebounding vein, and thus the length of

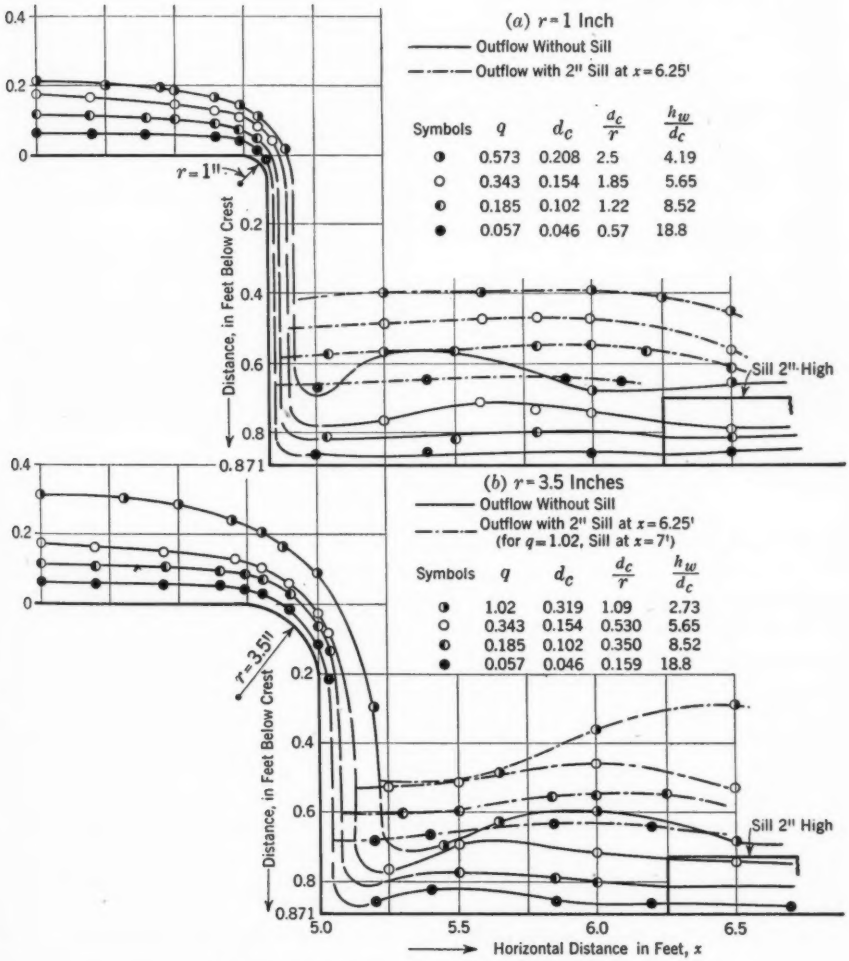

(a) r = 1 Inch
Outflow Without Sill
Outflow with 2" Sill at x = 6.25'
Symbols q $d_c$ $\frac{d_c}{r}$ $\frac{h_w}{d_c}$
0.573 0.208 2.5 4.19
0.343 0.154 1.85 5.65
0.185 0.102 1.22 8.52
0.057 0.046 0.57 18.8
r = 1"
Distance, in Feet Below Crest
Sill 2" High
(b) r = 3.5 Inches
Outflow Without Sill
Outflow with 2" Sill at x = 6.25'
(for q = 1.02, Sill at x = 7')
Symbols q $d_c$ $\frac{d_c}{r}$ $\frac{h_w}{d_c}$
1.02 0.319 1.09 2.73
0.343 0.154 0.530 5.65
0.185 0.102 0.350 8.52
0.057 0.046 0.159 18.8
r = 3.5"
Distance, in Feet Below Crest
Sill 2" High
Horizontal Distance in Feet, x

FIG. 20.—SURFACE PROFILES WITH AND WITHOUT SILL

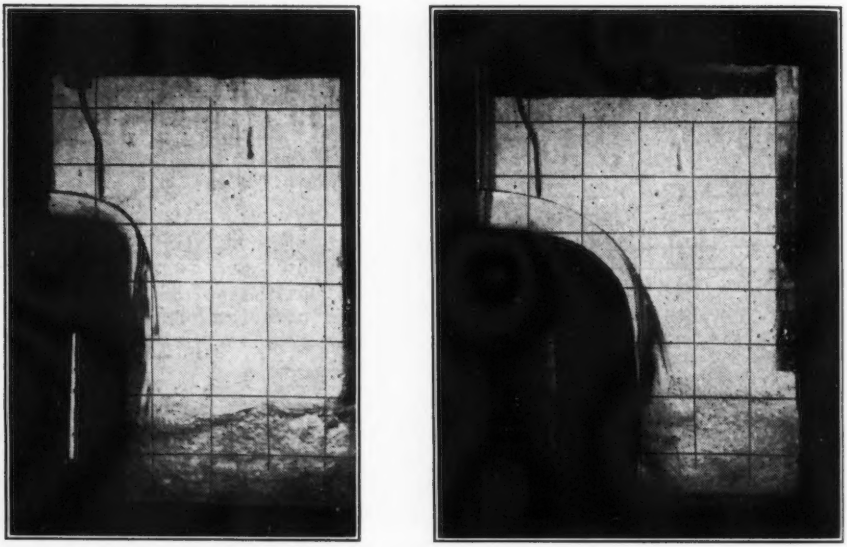
[20] Prof., Civ. Eng., Columbia Univ., New York, N. Y.

[21] Research Associate, Civ. Eng. Dept., Columbia Univ., New York, N. Y.

[21a] Received by the Secretary July 9, 1942.

the downstream channel required for the flow to assume more or less normal features.

It stands to reason that economy in construction, as well as more favorable outflow, would be secured if the falling vein were caused to strike the apron close



(*a*) $r = 1$ in.; and $q = 0.343$ cu ft per sec (*b*) $r = 3.5$ in.; and $q = 0.343$ cu ft per sec

FIG. 21.—VIEWS OF FLOW CHARACTERISTICS WITH A TWO-INCH SILL

to the wall and thus at a larger angle. This could be achieved by resorting to the so-called "adherent" type of nappes, which would follow the surface of the wall and which would strike the floor at the very toe of the structure. The general disadvantage of such nappes, of course, is their instability. In fact, when the brink of the fall has a sharp wedge, as in Figs. 3 and 6, the nappe, except for extremely low heads, will tend to detach itself from the wall, changing to the customary outline.

The situation can be remedied, however, by properly rounding the exit wedge. Indeed, observations show that, if the brink of the fall is shaped by inserting an arc of a circle, the adherent nappes become quite stable. Even a relatively small rounding seems to serve the purpose in securing a dependable vertical falling nappe, and that at comparatively high rates of flow. The situation is illustrated by Fig. 20, which depicts flow profiles obtained in the Fluid Mechanics Laboratory at Columbia University in New York, N. Y. The apparatus used in the experimenting has been well described elsewhere.[22] In Fig. 20(*a*) the wedge is rounded to a radius $r = 1$ in., whereas, in Fig. 20(*b*), $r = 3\frac{1}{2}$ in. In both figures the solid lines show the downstream surface profiles in free undisturbed outflow. The broken lines demonstrate the eventual effect

[22] *Proceedings*, Am. Soc. C. E., April, 1942, p. 637.

of an interposed 2-in. sill forming a stilling basin. The unit width discharges are reflected in the parametric values of the critical depth $d_c$, and the last two columns present the dimensionless ratios $\frac{d_c}{r}$ and $\frac{h_w}{d_c}$, $h_w$ being the height of the fall. Obviously, the foregoing ratios govern the flow forms. No exhaustive studies were made to establish the exact relationship at which the adherent nappe changes its form. In both instances, however, the profiles as plotted were close to the safe limit, as further increases in the flow brought about signs of instability.

A characteristic feature is the comparatively small rebounding effect accompanying vertically dropping veins, especially in the presence of a sill. Also, in the latter case, the outflow into the downstream channel was relatively calm, except in the instance of $q = 1.02$ cu ft per sec per ft of width in Fig. 20(*b*), where the size of the stilling pool was obviously insufficient. In fact, the general impression is that, with adherent vertical nappes, the quieting effect exercised by the stilling pool is more decisive than with detached nappes. The general physical aspects are partly illustrated by Fig. 21. The underpressures developed next to the rounded surface of the wedge were relatively small.

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DISCUSSIONS

# THE GREASE PROBLEM IN SEWAGE TREATMENT

## Discussion

BY MESSRS. DENIS DICKINSON, AND CHARLES R. VELZY

DENIS DICKINSON,[13] ESQ.[13a]—The question as to which solvent should be used for the determination of grease in sewage is associated closely with the definition of "grease." If the determination is made in order to estimate the difficulties that may arise in the purification of the sewage and the subsequent disposal of the sludge, then there would appear to be sufficient reason for the adoption of chloroform as the solvent, since it is affirmed that this liquid also extracts resinous substances that are likely to be more difficult to break down biologically than the less complex fatty acids. Especially does this apply to those cases in which disposal is left to auto-purification in the sea or in a water course. On the same grounds, chloroform should be adopted for testing the efficiency of any grease removal plant. On the other hand, if it is desired to assess the practicability of grease recovery, then a more selective solvent is preferable.

It is interesting to note that the ratio of the grease as determined by chloroform to that determined by petroleum ether is frequently constant for sludges from any one sewage, although the data available are not sufficient to justify any generalization on this point. It would appear that there is much to be learned from a study of the characteristics of the greases extracted by the various solvents.

The recovery of grease has been the practice at several works in England for many years, and there are other cases where floating fat is skimmed off (usually by scoops), drained, and sold for the manufacture of wagon grease, etc. The process for the recovery of grease by solvent extraction of the sludge has a further advantage in that the sludge is finally grease-free and weed-free, all seeds having been destroyed during the extraction process.[14] The latter

NOTE.—This paper by Almon L. Fales and Samuel A. Greeley, Members, Am. Soc. C. E., was published in February, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by Arthur D. Weston, M. Am. Soc. C. E.; May, 1942, by Harvey F. Ludwig, Jun. Am. Soc. C. E.; and June, 1942, by W. S. Mahlie, Esq.

[13] Campden, Gloucester, England; formerly Asst. Chemist, Stockport (England) Sewage Works.

[13a] Received by the Secretary June 2, 1942.

[14] "Borough of Halifax Sewage Disposal Works," anonymous, *Journal and Proceedings*, Inst. of Sewage Purification, 1937, I, p. 60.

is a very important point as it increases considerably the market value of the sludge as a fertilizer.

The practice of burying grease, which is mentioned in the paper, cannot be condemned too strongly, as it is to be feared that in many cases it merely defers the day on which final disposal must be undertaken.

A very large amount of grease is often discharged from slaughterhouses and meat products factories. Much of this can be eliminated by intelligent recovery at the factory. There is a tendency to make factory grease-traps too small; wherever possible they should be of a size such that the intermittent discharge of hot effluent cannot melt and emulsify the fat which normally floats on the surface at the outlet end of the trap.

CHARLES R. VELZY,[15] M. AM. SOC. C. E.[15a]—A comprehensive description of the general problems resulting from grease in sewage is presented in this paper, which includes methods of grease determination and removal, the effect upon receiving waterways, and a brief statement regarding methods of disposal. In sewage treatment works, grease disposal is always a problem to the operator. Its magnitude depends upon the size of the plant, the quantity and characteristics of the grease in the sewage, and the facilities of the treatment plant. Brief comments on the handling and disposal of grease at sewage treatment works are offered.

Although skimming of grease from sedimentation tanks is reasonably successful, the skimming equipment becomes coated with grease, it is unsightly, and there is a tendency to produce odors. The equipment is commonly cleaned by scraping. On larger installations special cleaning equipment using a combination of steam and a cleaning compound may be used effectively and with a saving of labor. Frequent cleaning will keep the equipment in satisfactory condition.

After grease has been removed from skimming tanks or sedimentation tanks, it must be transported to the point of disposal. In small plants this can be done by hand. In large plants pumping is probably the most satisfactory method. The difficulties involved result from the characteristic of grease to stick to whatever it comes in contact with, and from the presence of floating trash which comes off with the grease. This combination defies screening, tends to clog pipes and pump valves, and forms a stiff mat when allowed to stand.

Although these conditions cannot be eliminated, a number of things can be done to minimize the difficulties. Trash can be reduced by good screening of raw sewage. Grease can be kept more or less liquid by moving it frequently with sufficient water, using jets or sprays if necessary.

In some cases grease will coat the interior of piping through which it flows to such an extent as to increase friction head greatly and necessitate removal. This has occurred at the Buffalo, N. Y., plant and there the application of steam inside the sludge line has kept the line open. It has been necessary to apply steam at intervals of from four to six months. It has been introduced

[15] Asst. Engr., Greeley & Hansen, Buffalo, N. Y.

[15a] Received by the Secretary July 30, 1942.

through an injector with water so that the grease is actually melted and removed by hot water. Temperatures of approximately 180° F have been required.

The final disposal of grease will vary with the size of the treatment plant and other conditions. In the small plant where grease is handled manually it can be buried. In the moderate size plants it is sometimes piped to low areas where it is drained and burned. It is common practice to pump it with sludge to digestion tanks. This has the advantage of reducing the quantity, as some types will digest, and also adding to the amount of gas produced. Some grease, however, does not digest, or at least is very slow to digest, in which case it accumulates in the top of the digestion tank. If the accumulation becomes several feet thick, it tends to form a stiff mass, and takes up volume in the tank needed for digestion. This mass of floating material consists of trash with varying proportions of grease and is frequently called scum. If it is of excessive depth, it may become necessary to remove it. If the treatment plant employs incineration for sludge disposal, the obvious procedure is to burn the grease.

There are at least two problems involved in this method of disposal. One is that of handling the grease in this condition and the other is the control of temperature in the incinerator. At Buffalo the scum or grease in the sludge tanks is softened with a spray of supernatant liquor until it will flow and can be pumped. It is then drawn off and mixed with sludge for filtering and incineration. Temperature in the incinerator is controlled by regulating the the proportion of scum and sludge.

In general, problems of handling and disposing of grease must be solved at each sewage treatment plant according to the facilities of the plant and the characteristics of the grease obtained at the plant. Some procedures that have been used with varying degrees of success have been indicated. Grease in sewage, from the standpoints of determination, removal, handling, and disposal, constitutes one of the problems of sewage treatment requiring continued study and research.

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DISCUSSIONS

# TECHNIQUE OF DETERMINING SHEARING STRENGTH OF SOILS

## PROGRESS REPORT OF A SPECIAL COMMITTEE OF THE SOIL MECHANICS AND FOUNDATIONS DIVISION ON THE TECHNIQUE OF SOIL TESTS

### Discussion

BY PHILIP C. RUTLEDGE, ASSOC. M. AM. SOC. C. E.

PHILIP C. RUTLEDGE,[4] ASSOC. M. AM. SOC. C. E.[4a]—The committee has presented concisely the essential features of the several devices now in common use for the testing of soils in shear and has noted many of the problems still unsolved in their uses and in the applications of shear test results. The subject matter of the committee report is divided into three parts: (1) Equipment used for shear testing; (2) testing procedure including sample preparation; and (3) methods of interpreting and applying test data. The writer proposes to follow this division in his discussion.

*Equipment Used for Shear Testing.*—The committee has advisedly emphasized the merits and limitations of each type of shear testing device on the basis of the distribution of stresses within the specimens being tested. Consideration of stress distribution within shear test specimens can well be separated into two substantially independent parts:

(1) The distribution of stresses imposed on the entire soil mass of the test specimen by the mechanical features of the testing device; and

(2) The distribution of stresses within the soil mass between its solid, liquid, and gaseous phases.

The second of these two parts involves study of the true "intergranular stresses" referred to by the committee and is primarily a problem in testing procedure and interpretation of test results. The first part is specifically concerned with

NOTE.—This Report was published in February, 1942, *Proceedings*. Discussion on this Report has appeared in *Proceedings*, as follows: February, 1942, by Glennon Gilboy, Assoc. M. Am. Soc. C. E.; April, 1942, by Cloyd D. Beerup, Assoc. M. Am. Soc. C. E.; and June, 1942, by Jacob Feld, M. Am. Soc. C. E.

[4] Prof., Soil Mechanics, School of Civ. Eng., Purdue Univ., West Lafayette, Ind.

[4a] Received by the Secretary June 8, 1942.

the testing devices, which the committee has separated generically into translatory shear and cylindrical or triaxial compression types. An important feature in the distribution of stresses in each of these types has not been mentioned by the committee.

In the first place, translatory shear devices attempt to measure directly the tangential or shearing stresses acting along one or two planes within the test specimen. Shearing stresses result from the shearing strains induced by translatory motion between adjacent rigid parts of the device holding the specimen. Shearing strains are angular distortions of material within a zone of shear of finite thickness. For a given translatory motion between two parts of a specimen, shearing strains will vary directly with the thickness of the zone of shear. In round or rectangular box-type translatory devices the thickness of the zone of shear is controlled at the ends of the specimen by the space between the adjacent rigid translatory parts of the device and in the interior or the specimen by its over-all dimensions. If the specimen is thicker than the opening between the rigid parts of the device holding it, the thickness of the zone of shear will be greater in the interior than at the ends, and shearing strains and stresses will be nonuniform over the cross section in the direction of translation. Therefore progressive failure, starting at points of minimum thickness of the zone of shear, will occur in all except perfectly plastic materials. This characteristic of translatory devices was pointed out by D. W. Taylor,[5,6] Assoc. M. Am. Soc. C. E., in 1938.

In reference to cylindrical or triaxial compression testing devices, the committee has emphasized the unknown effect on internal stresses of the lateral restraint of friction at the rigid end-bearing surfaces through which axial load is applied to the specimen. Lateral end restraint is the common condition of all tension and compression tests on all engineering materials. Within the range of elastic deformations, Saint-Venant's principle states that "if the forces acting on a small portion of the surface of an elastic body are replaced by another statically equivalent system of forces acting on the same portion of the surface, this redistribution of loading produces substantial changes in the stress locally but has a negligible effect on the stresses at distances which are large in comparison with the linear dimensions of the surface on which the forces are changed."[7] Therefore, if a specimen is relatively long compared with its end diameter, lateral end restraint together with uneven application of the axial loads will cause negligible deviations from uniform stress in its central portion, where failure usually occurs. The length-to-diameter ratio of compression specimens is limited approximately to four by the tendency toward buckling of slender specimens. Thus, the distance from the ends to the zone of failure is limited "in comparison with the linear dimensions of the surface on which the forces are changed." This limitation is offset to some extent by more rapid redistribution toward statically equivalent uniform stress during semi-plastic deformations, because large strains cause relatively small

[5] "Notes on Soil Mechanics," by D. W. Taylor, Mass. Inst. Tech., revised by D. W. Taylor, 1938, Cambridge, Mass.

[6] "Shearing Properties of Ottawa Sand," by D. W. Taylor and T. M. Leps, *Proceedings*, Soils and Foundations Conference of the U. S. Engr. Dept., Boston, Mass., 1938.

[7] "Theory of Elasticity," by S. Timoshenko, New York, N. Y., 1934, p. 31.

changes in stress. For these reasons the writer cannot share the concern of the committee about nonuniform distribution of stresses in cylindrical compression specimens which have a length-to-diameter ratio as great as three.

Finally, in regard to testing devices, the committee deplores loading equipment which is "distinctly amateurish in appearance and construction" consisting of "water cans, wires, bicycle wheels, hanging weights, levers, pulleys, and wooden framework * * *." Would it not be better to judge testing equipment on the bases of its usability and of the accuracy and reliability of the results it produces, and not on its appearance?

*Testing Procedure.*—Discussion of testing procedure in the committee's report is confined to descriptions of "constant stress" and "constant strain" types of loading equipment and of the "quick," "preconsolidated quick," and "slow" shear tests of cohesive soils. The writer believes that testing procedure is fully as important as the testing device used. In tests on cohesive soils the results are almost completely dependent on the testing procedure and the condition of the specimens being tested. Such tests can be divided into two general groups:

(1) Tests on "undisturbed" specimens to determine the strength of the soil as it exists in nature; and

(2) Tests on remolded and compacted specimens to determine the strength of the soil after placing in an earth dam, levee, embankment, or foundation fill.

The results of unconfined compression tests, "quick" triaxial compression tests, and translatory shear tests under zero normal pressure on natural cohesive soils are seriously affected by sample disturbance. Relatively slight sample disturbance may reduce the strength obtained from tests to less than half of the actual strength in the ground. On the other hand, sample disturbance prior to "preconsolidated quick" and "slow" triaxial compression tests and translatory shear tests under normal pressures of one or more tons per square foot usually will result in the test being performed on a specimen with a void ratio considerably smaller than the natural void ratio of the soil in the ground. The result is a test on a disturbed specimen in a density condition quite different from any which will be attained by the soil under field loading conditions. Although the effects on shearing strength of sample disturbance and of decreased void ratio or increased density due to consolidation following sample disturbance are compensating, any approximation between test and natural strengths is accidental.

Tests on compacted soils to be used in the construction of earth dams or fills frequently have been performed on the material as compacted at or near the optimum water content. In this condition the soil has between 80% and 90% of its voids filled with water. F. H. Newnam, Jr., Assoc. M. Am. Soc. C. E., has shown that the air voids in terms of total volume of cohesive soil compacted at the standard Proctor optimum density will fall between 4% and 7%.[8] Tests on cohesive soils in this condition are not indicative of the strength of such materials in field applications in which they may become saturated.

[8] "Utilizing Available Soils in Airport Construction," by F. H. Newnam, Jr., *Proceedings*, Fifth Texas Conference on Soil Mechanics and Foundation Eng., Univ. of Texas, Austin, Tex., Pt. II, February, 1942.

In the opinion of the writer such specimens should be tested, not as compacted, but as nearly as possible in the condition in which they will exist after saturation under the field conditions. In other words, the specimens should be compacted, should be saturated under external stress conditions and water pressures which are approximately comparable to those which will occur in the prototype, and then tested. The results of such tests performed by the writer show smaller strengths and strength characteristics entirely different from those of the soil as initially compacted. In general, most such test specimens decrease in volume during saturation. The only exceptions have been specimens compacted in a relatively dry condition to densities greater than the standard Proctor optimum compaction and saturated under external normal loads less than 0.5 ton per sq ft. The maximum observed expansion in the latter group of specimens was 2.2% by total volume. Saturation under prototype conditions prior to testing eliminates the dangers, true or imaginary, of over-compaction of cohesive soils because strengths from tests on partly saturated specimens do not create a false sense of security in the mind of the designer.

*Interpreting and Applying Shear Test Data.*—Although the committee has stressed the importance of interpretation and application of the data obtained from shear tests, it has justifiably refrained from a detailed discussion of this subject, particularly as applied to cohesive soils. The results of shear tests must eventually be reduced to the Coulomb equation or a simple modification of it. Otherwise the complications introduced when shearing strength is applied in an actual stability problem become intolerable. Therefore interpretation of shear tests resolves itself into a reduction of the results, which are usually complicated by the effects of pore water stresses and sample disturbance, into values which can be substituted in some form of the Coulomb equation with reasonable accuracy for the range of probable prototype stresses.

The committee's report gives the impression that analyses of stresses in the prototype are a vital factor in the application of the results of tests for the shearing strength of soil. In the opinion of the writer, the probabilities of even crudely approximate analyses of mass stresses in soil in which the actual stresses approach the shearing strength are remote. The solutions of theory of elasticity, which are practically the only tools now available for the approximation of soil stresses, are limited to the range of reasonably elastic behavior of the soil. For most soils this range does not extend beyond 50% of the soil strength and the deformations caused by stresses corresponding to the last 50% of the soil strength may easily be ten times those for the first 50%. Satisfactory solutions for problems in which the soil shearing strength is the controlling factor depend on knowledge of the shape and position of the surface along which shearing rupture will take place. If both the shape and position of a rupture surface are known from field observations, the problem becomes statically determinate. Reference to stresses and deformations can then be confined to secondary considerations such as the determination of the points of application of external reactions on the sliding soil mass. The distinction here is parallel to the basic difference between elastic analysis and limit design in structural engineering. In many of the available stability solutions only

the shape of the most probable rupture surface is known and repeated statically determinate solutions are made until the position for the surface of the given shape is found which results in a minimum resistance to rupture.

The essential point in applications of shear test results to studies of safety against failure is that analyses of mass stresses and field observations of mass stresses in prototypes are of little value unless the soil is considered, or actually is, on the point of failure. Field observations of the mode of failure, of the actual shapes of the surfaces along which shearing rupture takes place in nature, are the essence of such applications. This qualification of the committee's report refers exclusively to mass stresses, as opposed to intergranular and pore-water stresses. Field observations of pore-water stresses in prototypes are tremendously important but are beyond the scope of the points the writer wishes to make.

The significant feature of the Swedish circle method for the analysis of stability of slopes, which was mentioned by the committee, is that it was based on, and has been confirmed by, field observations of the shapes of actual sliding surfaces in slopes. This feature is completely independent of shear test results or the way in which the test results are applied to a statical analysis. Any improvement in testing equipment or procedure, or in interpretation of test results, is an improvement in the solutions obtained by this method. The method itself can be improved only by possibly more precise approximations of actual sliding surfaces.

The writer is in full agreement with the conclusions of the committee. In particular, conclusion (*e*), with the concluding remarks concerning soil stresses interpreted to mean conditions of failure in prototypes, deserves careful thought by every engineer engaged in soils studies.

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DISCUSSIONS

# VISCOSITY AND SURFACE TENSION EFFECTS ON V-NOTCH WEIR COEFFICIENTS

## Discussion

BY MESSRS. GLEN N. COX AND F. J. GERMANO, EMERY H. WILLES, AND RALPH W. POWELL

GLEN N. COX,[7] M. AM. SOC. C. E., AND F. J. GERMANO,[8] ASSOC. M. AM. SOC. C. E.[8a]—It is indeed refreshing to read a paper such as that presented by Professor Lenz. With this paper, the explored realm of fluid mechanics and dimensional analysis has been widened, and the author is to be congratulated on the capable manner in which this has been accomplished.

Since the publication of Reynolds' outstanding paper,[9] a number of papers have been presented on the flow in pipes in which an attempt has been made to obtain a wider correlation and to make the results of more general application. Although further advance has been, and will be, made in this connection, the paper by Kemler[10] stands out as the pioneering attempt to make the results of general use to the engineering profession. The solution of the practical pipe problems is now commonly made by means of these generalized pipe friction coefficients.

As the generalized method became more commonly accepted as desirable for the solution of practical pipe-flow problems, attention turned to obtaining generalized coefficients for the various measuring devices. Excellent generalizations of the coefficients for venturi tubes[11] and orifice meters[12] have been presented. The current paper extends the same type of reasoning to the coefficients of the V-notch weir and covers the widest range of conditions con-

NOTE.—This paper by Arno T. Lenz, Assoc. M. Am. Soc. C. E., was published in March, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: June, 1942, by A. A. Kalinske, Assoc. M. Am. Soc. C. E.

[7] Prof., Mechanics and Hydraulics, Louisiana State Univ., Baton Rouge, La.

[8] Asst. Prof., Eng. Mechanics, Louisiana State Univ., Baton Rouge, La.

[8a] Received by the Secretary June 29, 1942.

[9] "An Experimental Investigation of the Circumstances Which Determine Whether the Motion of Water Shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels," by Osborne Reynolds, *Philosophical Transactions*, Royal Soc. of London, Vol. 174, Pt. III, 1883.

[10] "A Study of the Data on the Flow of Fluids in Pipes," by Emory Kemler, *Transactions*, A. S. M. E., Vol. 55, 1933, HYD-55-2, p. 7.

[11] "Fluid Meters—Their Theory and Applications," *Research Publication*, A. S. M. E., 4th Ed., 1937.

[12] "Orifice Discharge Coefficients for Viscous Liquids," by G. L. Tuve and R. E. Sprenkle, *Instruments*, Vol. 6, 1933, pp. 201–206; and Vol. 8, 1935, pp. 202–205, 225, and 232–234.

tained in any of the papers on weirs which have been published to date. Although certain refinements and additions may follow, the paper must be considered to be outstanding in assisting in the extension of the generalized reasoning.

Eq. 13 does not follow as obviously from Eqs. 11 and 12 as might be expected from a cursory reading of the explanatory paragraph. Eqs. 11 and 12 are those of the projections on the *C*–**R** and *C*–**W** planes of a line in the plane represented by Eq. 13. Such a plane is illustrated as *G H I* in Fig. 18. Line *A B* is a particular line in this plane. For a given angle of notch, the experimental data plot as a series of parallel lines in the same plane, one line for each fluid.

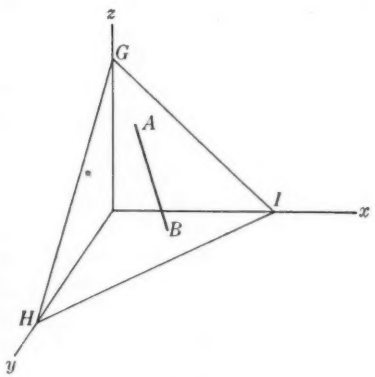


FIG. 18

Since the exponents, *m* and *n*, of Eqs. 11 and 12 do not refer to the same things as those in Eq. 13, confusion might have been avoided by using letters in the equations of the projections of the lines differing from those relating to the plane.

It would increase the value of the paper if Professor Lenz would prepare a three-dimensional figure showing the plane on which the line (Eq. 13) appears and showing the projections of this line that give Eqs. 11 and 12. The values of *m* and *n* from Eqs. 11, 12, and 13 should be indicated, as well as the three values of *B* for these equations. This could be done for one angle of the notch —say 45°. To do so effectively, at least two experimental curves must be drawn.

It would appear that undue emphasis has been placed upon the failure of Eq. 3 to apply for the full range in the values of **R** and **W**. Experience has taught engineers that certain factors predominate over a certain range, whereas others may govern over another. Thus, there are three definite ranges for the coefficient of drag for the flow past a cylinder, or for the value of *f* for the flow through pipes. There are also three ranges for the value of *C* for the flow through a pipe orifice, but there are only two ranges for the value of *C* for the flow through a venturi tube. There are three ranges for the coefficient for the V-notch weir—namely, that for which the flow is primarily governed by the surface tension of the liquid, that for which it is primarily governed by the viscosity of the liquid, and then that for which it is independent of the surface tension and the viscous effects. It seems rather unreasonable to expect one empirical equation (even though derived rationally) to cover the various ranges without loss in accuracy. It should suffice simply to state the applicable range without the need for a feeling of apology.

EMERY H. WILLES,[13] JUN. AM. SOC. C. E.[13a]—In the third paragraph of the "Synopsis" the author makes the statement: "It was found that the weir

[13] Hydrographer, Utah Copper Co., Salt Lake City, Utah.

[13a] Received by the Secretary July 2, 1942.

coefficient increases with both viscosity and surface tension, and that it decreases with increase in head and angle of V-notch." If an increase of viscosity and surface tension does increase the weir coefficient, it naturally would follow that an increase in head and angle of V-notch would decrease the weir coefficient, for as the head and angle of V-notch increases the effect of viscosity and surface tension decreases, and hence the weir coefficient decreases. Is it true that the weir coefficient will increase with viscosity and surface tension? This statement is just the opposite from that which the writer would expect, for viscosity is the reciprocal and not a measure of fluidity. If the author's statement holds true for highly viscous liquids, then, for a given head, glucose, honey, or molasses would flow faster than water. The writer noticed for the oils used (with the exception of oil C) that at a given temperature an oil with a higher viscosity had a lower surface tension (see Fig. 19). This might indicate that the effect of decreasing surface tension was to overcome the effect due to the increase in viscosity; but this is contrary to the author's statement ("Analysis of Data: Development of General Equation"): "Furthermore, it is generally supposed that viscosity plays a much more important rôle than surface tension in determining $C$."

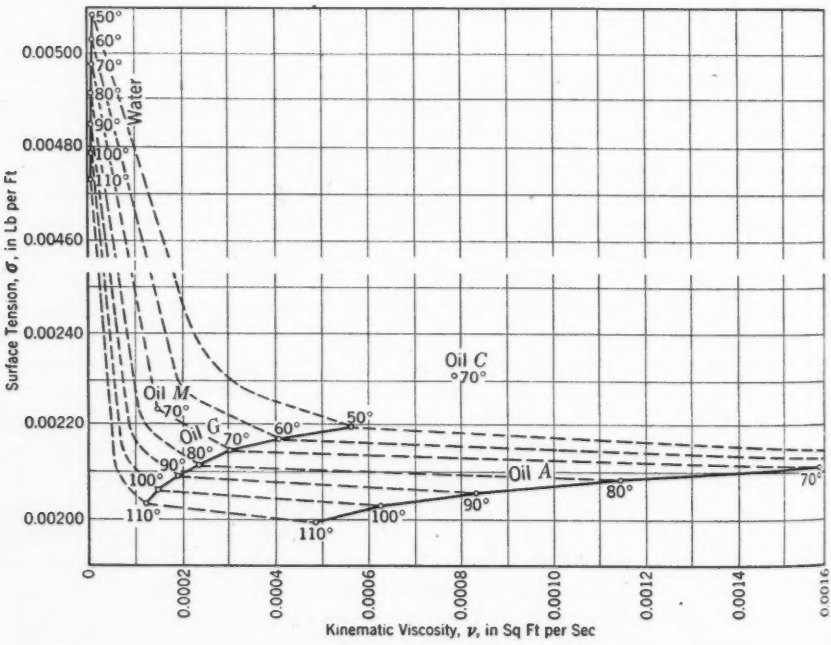


FIG. 19.—RELATIONSHIP BETWEEN VISCOSITY AND SURFACE TENSION FOR THE GIVEN OILS

If the same oil is used and the temperature varies (Fig. 12), the surface tension increases with the viscosity. This relationship is just the reverse of that noticed for different oils at the same temperature. These two opposite relationships, together with the author's experimental results, bear out the

author's statement that "viscosity plays a much more important rôle than surface tension," and prove the falsity of the writer's suggestion herein.

What, then, is the answer to the apparent reversal of natural laws? In Fig. 13, for values of Reynolds' number greater than a certain critical value (for example, $\mathbf{R}_{cr} = 845$ for a 28° V-notch) the plotted points fall on the author's straight line; but for values of Reynolds' number less than this critical value there is a rapid drawing away from the straight line, apparently forming a parabolic curve. Thus, for the lower values of Reynolds' number—or, in other words, as the viscosity increases, or becomes more important—the coefficient ($C$) will decrease instead of increase. At some value of Reynolds' number the tangent to the true curve becomes zero ($\mathbf{R}$ = approximately 400 for a 28° V-notch). For values of Reynolds' number less than this the slope of the tangent is positive and for values of Reynolds' number greater than this the slope is negative. Thus, for the highly viscous liquids the coefficient will decrease as the viscosity increases, which would be expected.

In 1643 Torricelli in his *De Motu Gravium Naturaliter Accelerato* enunciated the law that is the base of all weir formulas—namely:

> "The velocity of a fluid passing through an orifice in the side of a reservoir is the same as that which would be acquired by a heavy body falling freely through the vertical height measured from the surface of the fluid in the reservoir to the center of the orifice."

By the use of Torricelli's law and calculus, Eqs. 1 and 2 are easily derived. This derivation is simple and may be found in most hydraulic textbooks. The author arrives at the same formula using a method developed by Edgar Buckingham[14] in 1915—namely, the "$\pi$-theorem." This method of dimensional analysis, because its value has been proved in model experiments in hydraulic laboratories, is now often used. Because of this increase in importance, a detailed investigation of the development of this theorem may be in order.

If the physical quantities, $G_1, G_2, \ldots G_n$, such as a length, a force, a velocity, a density, etc., suffice to describe any physical phenomenon completely, then they are mutually dependent in some known or unknown manner. This mutual dependence may be stated mathematically as a function of $n$ independent variables:

$$f(G_1, G_2, \ldots G_n) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

In other words, the mutual dependence of the $n$ quantities can be expressed as a polynomial in $n$ independent variables. Any polynomial of these $n$ variables may be written:

$$K_1 (G^{a_1})_1 (G^{b_1})_2 \ldots (G^{m_1})_n + K_2 (G^{a_2})_1 (G^{b_2})_2 \ldots (G^{m_2})_n + \ldots K_n (G^{a_n})_1 (G^{b_n})_2 \ldots (G^{m_n})_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

or

$$\Sigma(K G^{a_1} G^{b_2} \ldots G^{m_n}) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

in which $K$ is a dimensionless coefficient and $a, b, \ldots m$ are exponents.

[14] "Model Experiments and the Forms of Empirical Equations," by E. Buckingham, *Transactions*, A. S. M. E., Vol. 37, 1915, p. 289.

If Eq. 22 is divided by one of the terms of Eq. 21—$K_i\ (G^{a_i})_1\ (G^{b_i})_2\ \ldots\ (G^{m_i})_n$:

$$\Sigma(J\ G^{\alpha}{}_1\ G^{\beta}{}_2\ \ldots\ G^{\mu}{}_n) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (23a)$$

in which $J = \frac{K}{K_i}$, $\alpha = \frac{a}{a_i}$, $\beta = \frac{b}{b_i}$, $\ldots\ \mu = \frac{m}{m_i}$; or, since one of the terms is equal to unity:

$$1 + \Sigma(J\ G^{\alpha}{}_1\ G^{\beta}{}_2\ \ldots\ G^{\mu}{}_n) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (23b)$$

In his "Deduction of the $\pi$-Theorem"[14] Buckingham makes the statement:

> "Now all the terms of any physical equation must have the same dimensions and the coefficients $[J]$ have no dimensions because they are merely ratios of the dimensionless coefficients $[K]$."

This is merely another way of expressing one of the first principles of physical mathematics—namely, dogs, cats, and pigs cannot physically be added together; they must be restated in terms of some common attribute. In other words, each term of Eq. 21 or Eq. 22 must have the same dimensions. Upon the validity of this statement depends the proof of the theorem and hence the usefulness of dimensional analysis. Since each term in Eq. 21 or Eq. 22 must have the same dimensions, each term in Eq. 23 will be dimensionless.

Every physical quantity is composed of two distinct characteristics—magnitude and dimension. Magnitude is dimensionless and may be represented by a numerical constant $K$, whereas the dimension of the quantity is determined by certain fundamental units such as mass, length, time, etc., to some power which is known or can be determined as soon as the quantity itself is known. In mechanics the number of these fundamental units is three—mass, length, and time. In problems dealing with thermal and electromagnetic as well as mechanical quantities, the number of fundamental units will not exceed five. The dimension of a quantity may be represented, therefore, by $M^x$, $L^y$, $T^z$, $\ldots\ D^n{}_k$, in which $k$ represents the number of fundamental units and $M$, $L$, $T$, and $D$ are symbols for these units. Thus,

$$G = K\ M^x\ L^y\ T^z\ \ldots\ D^n{}_k \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Assuming for simplicity that mechanical quantities only are involved, Eq. 23 becomes,

$$1 + \Sigma\ [J\ K^{\alpha}{}_{G1}\ K^{\beta}{}_{G2}\ \ldots\ K^{\mu}{}_{Gn}\ (M^{x_1}\ L^{y_1}\ T^{z_1})^{\alpha}\ (M^{x_2}\ L^{y_2}\ T^{z_2})^{\beta}$$
$$\ldots\ (M^{x_n}\ L^{y_n}\ T^{z_n})^{\mu}] = 0 \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

As noted previously, Eq. 23 is dimensionless; hence,

$$(M^{x_1}\ L^{y_1}\ T^{z_1})^{\alpha}\ (M^{x_2}\ L^{y_2}\ T^{z_2})^{\beta}\ \ldots\ (M^{x_n}\ L^{y_n}\ T^{z_n})^{\mu} = [1] \ldots\ldots (26)$$

in which $[1]$ represents the expression which has the same dimension as unity. Eq. 25, therefore, becomes,

$$1 + \Sigma[J\ K^{\alpha}{}_{G1}\ K^{\beta}{}_{G2}\ \ldots\ K^{\mu}{}_{Gn}] = 0 \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

and from Eq. 26:

$$\left.\begin{aligned} \alpha\ x_1 + \beta\ x_2 + \ldots\ \mu\ x_n &= 0 \\ \alpha\ y_1 + \beta\ y_2 + \ldots\ \mu\ y_n &= 0 \\ \alpha\ z_1 + \beta\ z_2 + \ldots\ \mu\ z_n &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

The unknowns in Eqs. 28 are $\alpha, \beta, \ldots \mu$. The remainder will be known in any particular problem.

Again referring to Buckingham's paper:[15]

> "If $k$ is the number of fundamental units needed, there is always, among all the $n$ [quantities], at least one set of $k$ which are independent of one another and might themselves be used as fundamental units if we paid no attention to the question of representing and preserving the fundamental units by satisfactory primary standards."

Letting $G_1, G_2, \ldots G_k$ represent the chosen quantities in Eq. 20 that may be used as dimensions and $P_1, P_2, \ldots P_{n-k}$ be the quantities that are left, then from the foregoing statement:

$$[P_1] = [G^a{}_1 G^b{}_2 \ldots G^i{}_k] \ldots\ldots\ldots\ldots\ldots\ldots (29a)$$

in which the brackets are used as a symbol of "the dimension of." Substituting the dimensions of Eq. 24 in Eq. 29$a$, then:

$$M^{x_p} L^{y_p} T^{z_p} \ldots (D^{n_p})_k = [M^{x_1} L^{y_1} T^{z_1} \ldots (D^{n_1})_k]^a [M^{x_2} L^{y_2} T^{z_2} \ldots (D^{n_2})_k]^b \ldots [M^{x_k} L^{y_k} T^{z_k} \ldots (D^{n_k})_k]^i \ldots\ldots\ldots (29b)$$

Letting $\pi$ equal the ratio of the terms on both sides of the equal sign in Eqs. 29:

$$\pi = G^A{}_1 G^B{}_2 \ldots G^I{}_k P \ldots\ldots\ldots\ldots\ldots\ldots (30a)$$

in which $A = -a$, $B = -b$, etc. Also,

$$\pi = [M^{x_1} L^{y_1} T^{z_1} \ldots (D^{n_1})_k]^A [M^{x_2} L^{y_2} T^{z_2} \ldots (D^{n_2})_k]^B \ldots [M^{x_k} L^{y_k} T^{z_k} \ldots (D^{n_k})_k]^I M^{x_p} L^{y_p} T^{z_p} \ldots (D^{n_p})_k = [1] \ldots (30b)$$

in which the brackets are again used as a symbol of "the dimension of." As demonstrated in Eqs. 28, it is possible to write from Eq. 30$b$:

$$\left.\begin{array}{l} A_{x_1} + B_{x_2} + \ldots I_{x_k} + x_p = 0 \\ A_{y_1} + B_{y_2} + \ldots I_{y_k} + y_p = 0 \\ A_{z_1} + B_{z_2} + \ldots I_{z_k} + z_p = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{n_1} + B_{n_2} + \ldots I_{n_k} + n_p = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (31)$$

As in Eqs. 28, the unknowns are $A, B, \ldots I$ and, since there are as many equations as there are unknowns, Eqs. 31 can be solved. Hence, Eqs. 30 are completely solved and Eq. 27 becomes:

$$1 + \Sigma[J (\pi^{r_1})_1 (\pi^{r_2})_2 (\pi^{r_3})_3 \ldots (\pi^{r_{n-k}})_{n-k}] = 0 \ldots\ldots\ldots (32a)$$

The only unknowns now are the value of the exponents "$r$" for each $\pi$-value and each term. Since the "1" in Eq. 32$a$ is also one of the terms of Eq. 22 divided by itself, it also may be included in the summation, and therefore

[15] "Model Experiments and the Forms of Empirical Equations," by E. Buckingham, *Transactions*, A. S. M. E., Vol. 37, 1915, p. 290.

Eq. 32*a* can be written:

$$\Sigma[J\ (\pi^{r1})_1\ (\pi^{r2})_2\ (\pi^{r3})_3\ \ldots\ (\pi^{r_{n-k}})_{n-k}] = 0 \quad \ldots\ldots\ldots (32b)$$

or

$$f(\pi_1, \pi_2\ \ldots\ \pi_{n-k}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (32c)$$

which is the same as Eq. 4*b* of the paper.

By choosing $V$, $H$, and $\rho$ as the independent quantities to be used as fundamental units:

$$\left.\begin{aligned} \pi_1 &= V^{A_1} H^{B_1} \rho^{C_1} \mu \\ &= (K^{A_1})_V\ (K^{B_1})_H\ (K^{C_1})_\rho\ K_\mu \left[ \left( \frac{L}{T} \right)^{A_1} L^{B_1} \left( \frac{F\ T^2}{L^4} \right)^{C_1} \left( \frac{F\ T}{L^2} \right) \right] \\ \pi_2 &= V^{A_2} H^{B_2} \rho^{C_2} \sigma \\ &= (K^{A_2})_V\ (K^{B_2})_H\ (K^{C_2})_\rho\ K_\sigma \left[ \left( \frac{L}{T} \right)^{A_2} L^{B_2} \left( \frac{F\ T^2}{L^4} \right)^{C_2} \left( \frac{F}{L} \right) \right] \\ \pi_3 &= V^{A_3} H^{B_3} \rho^{C_3} g \\ &= (K^{A_3})_V\ (K^{B_3})_H\ (K^{C_3})_\rho\ K_g \left[ \left( \frac{L}{T} \right)^{A_3} L^{B_3} \left( \frac{F\ T^2}{L^4} \right)^{C_3} \left( \frac{L}{T^2} \right) \right] \end{aligned}\right\} \ldots (33)$$

and from Eq. 6

$$\left.\begin{aligned} \left( \frac{L}{T} \right)^{A_1} L^{B_1} \left( \frac{F\ T^2}{L^4} \right)^{C_1} \left( \frac{F\ T}{L^2} \right) &= [1] \\ \left( \frac{L}{T} \right)^{A_2} L^{B_2} \left( \frac{F\ T^2}{L^4} \right)^{C_2} \left( \frac{F}{L} \right) &= [1] \\ \left( \frac{L}{T} \right)^{A_3} L^{B_3} \left( \frac{F\ T^2}{L^4} \right)^{C_3} \left( \frac{L}{T^2} \right) &= [1] \end{aligned}\right\} \ldots\ldots\ldots\ldots (34)$$

When the exponents in Eqs. 34 are determined:

$$\left.\begin{aligned} \pi_1 &= \frac{\mu}{V\ H\ \rho} = \frac{\nu}{V\ H} \\ \pi_2 &= \frac{\sigma}{V^2\ H\ \rho} \\ \pi_3 &= \frac{g\ H}{V^2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

When these values of $\pi$ are substituted in Eq. 32*c*, Eq. 5 is the result, from which the author develops Eq. 7.

To get Eq. 8 the author replaces "$V$ in each of the products by a dimensional equivalent, $\sqrt{g\ H}$." If the particular function of Eq. 7 were known, this substitution would not be entirely correct because the true relationship between $V$ and $g\ H$ is Eq. 6*a*; but, since the particular function is not known, this replacement is permissible. At least it is dimensionally correct. A more direct approach to Eq. 8 would be to choose $H$, $g$, and $\rho$ as the three units to

be used as dimensions instead of $V$, $H$, and $\rho$. Thus,

$$\left.\begin{aligned} \pi_1 &= H^{A_1}\, g^{B_1}\, \rho^{C_1}\, V \\ &= (K^{A_1})_H\, (K^{B_1})_g\, (K^{C_1})_\rho\, K_V \left[ L^{A_1} \left( \frac{L}{T^2} \right)^{B_1} \left( \frac{F\,T^2}{L^4} \right)^{C_1} \left( \frac{L}{T} \right) \right] \\ \pi_2 &= H^{A_2}\, g^{B_2}\, \rho^{C_2}\, \mu \\ &= (K^{A_2})_H\, (K^{B_2})_g\, (K^{C_2})_\rho\, K_\mu \left[ L^{A_2} \left( \frac{L}{T^2} \right)^{B_2} \left( \frac{F\,T^2}{L^4} \right)^{C_2} \left( \frac{F\,T}{L^2} \right) \right] \\ \pi_3 &= H^{A_3}\, g^{B_3}\, \rho^{C_3}\, \sigma \\ &= (K^{A_3})_H\, (K^{B_3})_g\, (K^{C_3})_\rho\, K_\sigma \left[ L^{A_3} \left( \frac{L}{T^2} \right)^{B_3} \left( \frac{F\,T^2}{L^4} \right)^{C_3} \left( \frac{F}{L} \right) \right] \end{aligned}\right\} \quad \ldots (36)$$

and solving:

$$\left.\begin{aligned} \pi_1 &= \frac{V}{\sqrt{g\,H}} \\ \pi_2 &= \frac{\mu}{H^{1.5}\, g^{0.5}\, \rho} = \frac{\nu}{H^{1.5}\, g^{0.5}} \\ \pi_3 &= \frac{\sigma}{H^2\, g\, \rho} \end{aligned}\right\} \quad \ldots (37)$$

Substituting these values of $\pi$ in Eq. 32*c* and solving for $C$, as demonstrated by the author, will yield Eq. 10.

The writer takes exception to the author's statement ("Analysis of Data: Development of General Equation"): "The functional relationship, $f$", between $C$, **R**, and **W** for a given weir must be obtained from experimental data." This statement should be: "The functional relationship * * * at present must be obtained from experimental data."

There is a theoretical solution to every phenomenon in nature. The trouble lies with Man. Either the relationship is so simple that Man has not looked in that direction yet, or it is so complex that at present it exceeds the capacity of the mind to solve it. The fact still remains, however, that there is a theoretical solution to the problem. Eq. 10 will be of the form

$$N_1\, (\pi^{n_1})_1\, (\pi^{m_1})_2 + N_2\, (\pi^{n_2})_1\, (\pi^{m_2})_2 + \ldots N_n\, (\pi^{n_n})_1\, (\pi^{m_n})_2 = 0 \quad \ldots (38)$$

Substituting $\pi_1 = \frac{1}{\mathbf{R}}$, $\pi_2 = \frac{1}{\mathbf{W}}$, and $N = B_x$, Eq. 38 becomes:

$$\frac{B_{x_1}}{\mathbf{R}^{n_1}\, \mathbf{W}^{m_1}} + \frac{B_{x_2}}{\mathbf{R}^{n_2}\, \mathbf{W}^{m_2}} + \ldots \frac{B_{x_n}}{\mathbf{R}^{n_n}\, \mathbf{W}^{m_n}} = 0 \quad \ldots (39)$$

or letting $C_x$ be the value of the function as $\pi_1$ and $\pi_2$ approach zero,

$$C - C_x = \frac{B_{x_1}}{\mathbf{R}^{n_1}\, \mathbf{W}^{m_1}} + \frac{B_{x_2}}{\mathbf{R}^{n_2}\, \mathbf{W}^{m_2}} + \ldots \frac{B_{x_n}}{\mathbf{R}^{n_n}\, \mathbf{W}^{m_n}} \quad \ldots (40)$$

or

$$C - C_x = \sum \frac{B_x}{\mathbf{R}^n\, \mathbf{W}^m} \quad \ldots (41)$$

If for simplicity all terms but one are assumed either zero or unimportant: $C - C_x = \frac{B_x}{\mathbf{R}^n \mathbf{W}^m}$ (see Eq. 13). If the effect of Weber's number ($\mathbf{W}$) is insignificant, Eq. 13 becomes $C - C_x = \frac{B_x}{\mathbf{R}^n}$ (compare Eq. 12). As noted previously, these equations can be used only between certain limits. Beyond these limits Eq. 41 must be used, for the experimental data when plotted on logarithmic paper seem to pull away from the author's straight line (see Fig. 13) and form a curve resembling a parabola. Just what the particular function of Eq. 41 is probably will be revealed or approximated by further experimental investigation.

RALPH W. POWELL,[16] M. AM. SOC. C. E.[16a]—This paper has been long desired, as it not only lays a basis for the measurement of viscous liquids by means of notch weirs, but also indicates the way in which the flow of less viscous fluids, like water, may be affected by changes of temperature. For many years the writer has felt that, instead of trying to study the effect of temperature upon the coefficient of notched weirs which measure water, by great refinement of measurement, it would be more satisfactory to find the influence of viscosity by the use of more viscous liquids where the effect would be greatly increased and therefore more readily measured. Mr. Lenz now shows that surface tension also has an important effect and gives quantitative results.

However, the writer feels constrained to mention several things that he personally wishes had been done differently. One minor point is that he prefers to define the Reynolds number of a weir as $\frac{V H}{\nu}$, in which $V$ is $\sqrt{2 g H}$, the ideal velocity due to the head $H$. This gives $\mathbf{R} = \frac{\sqrt{2 g}\, H^{1.5}}{\nu}$, or $\sqrt{2}$ times as much as Eq. 9*a*. Similarly, he would follow the definition of the Weber number given by Hunter Rouse,[17] Assoc. M. Am. Soc. C. E.—namely, $\frac{V}{\sqrt{\frac{\sigma}{\rho H}}}$, which reduces to $\mathbf{W} = \sqrt{\frac{2 g \rho}{\sigma}}\, H = H \sqrt{\frac{2 \gamma}{\sigma}}$. This is the square root of twice the value given by Eq. 9*b*. However, in the following comments, the author's definitions have been used. The writer also very much wishes that the actual numerical data had been included in the paper. Desiring to test the possibility of representing the author's experimental results by a formula of different form, he obtained a copy of the original thesis[4] from the University

[16] Associate Prof. of Mechanics, Ohio State Univ., Columbus, Ohio.

[16a] Received by the Secretary July 8, 1942.

[17] "Fluid Mechanics for Hydraulic Engineers," by Hunter Rouse, McGraw-Hill Book Co., Inc., New York, N. Y., 1938, p. 23.

[4] "The Effect of Viscosity and Surface Tension Upon V-Notch Weir Coefficients," by Arno T. Lenz; thesis presented to the University of Wisconsin in 1940, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

of Wisconsin, and was surprised to find that even this did not give the original measurements.

In 1910 T. P. Strickland[18] showed that the coefficient for water flowing through a V-notch weir could be very accurately represented over a wide range of heads by a formula of the form

$$C = a + \frac{b}{\sqrt{H}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

Scaling values from the curves of the paper and plotting $C$ against $\frac{1}{H}$ indicated that all the author's values for a 90° notch above the minimum heads shown in Fig. 7 fit this formula with a maximum discrepancy of about 0.5%. This is using $a = 0.552$ for all four liquids and the value of $b$ for each liquid that gives the best fit.

The writer wished to investigate a formula of the form

$$C = a + \frac{B_1}{\mathbf{R}^n} + \frac{B_2}{\mathbf{W}^m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43)$$

as an alternative to Eq. 3. Since $\mathbf{R}$ contains $H^{1.5}$ and $\mathbf{W}$ contains $H^2$, Eq. 42 would require that, in Eq. 43, $n = \frac{1}{3}$ and $m = \frac{1}{4}$. It was found that the best fit for a formula of this type and the author's data on 90° notch weirs was given by

$$C = 0.552 + \frac{0.435}{\mathbf{R}^{1/3}} + \frac{0.201}{\mathbf{W}^{1/4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44)$$

All the points shown in Fig. 7 (except those below the minimum head indicated) were computed by this formula and the discrepancies tabulated. The algebraic mean of the discrepancies was 0.0003 and the arithmetic mean, 0.0045. This is about the same mean discrepancy as the author's formula gives, and it seems to the writer that Eq. 44 is somewhat simpler than Eq. 3. At least it has the advantage of separating the viscosity and surface tension "corrections." It is interesting to note that, according to Eq. 44 for water, the effect of surface tension is nearly three and one-half times as great as that of viscosity. As is to be expected, for the oils the viscosity had more effect than the surface tension —nearly twice as much in the case of oil A at 70° F.

It should also be noted that, whereas the range of viscosities studied is quite large, the variation in surface tension is relatively small. The property that is effective here is what might be called the "kinematic surface tension," $\frac{\sigma}{\rho g}$ or $\frac{\sigma}{\gamma}$, where $\gamma$ is the specific weight. The expression $\frac{\sigma}{\gamma}$ is an area, say feet squared, and

$$\mathbf{W} = \frac{H^2}{\text{kinematic surface tension}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

When one consults a handbook for possible liquids that would give a wider range of kinematic surface tensions than those used by the author, the search

[18] *Engineering* (London), Vol. 90, October 28, 1910, p. 598.

is not very rewarding. Mercury has a surface tension some seven times as great as water, but it weighs 13.6 times as much, so that its kinematic surface tension is less—about the same as that of the oils used by the author. Water seems to stand at the top of the list (except that some water solutions of substances like $CaCl_2$ are somewhat higher). Carbon tetrachloride has a kinematic surface tension of about one fourth that of water and would probably be the best liquid to compare with water to get a more accurate measure of the effect of surface tension. Another liquid that might be used is water to which a small amount of saponin and a still smaller amount of oleic acid have been added.[19] This will not give nearly as wide a range as carbon tetrachloride (in fact a little less than the oil used by the author), but it would have the advantage of having the same viscosity as water alone, and therefore of varying only one factor at a time.

The writer has not taken time to develop formulas similar to Eq. 44 for other notch angles, and to study the variation of the coefficients with $\theta$. It would be only after such a study that the relative merits of a formula of the form of Eq. 44 could be compared with the author's Eqs. 3 and 14, and of course there may be some other form that is better than any of these.

The author has given little attention to the interesting question of the variation of $C$ with **R** and **W** when either or both of the latter are below what he calls the minimum values. Water is the best liquid with which to study low Weber numbers, and Barr's experiments ran down to a Weber number of about 54. These results, put in the units of this paper, and with the temperature assumed to be 46.4° F (the mean of the range given), are plotted in Fig. 20. (Some of these data were used by the writer[20] elsewhere; but there the coefficient

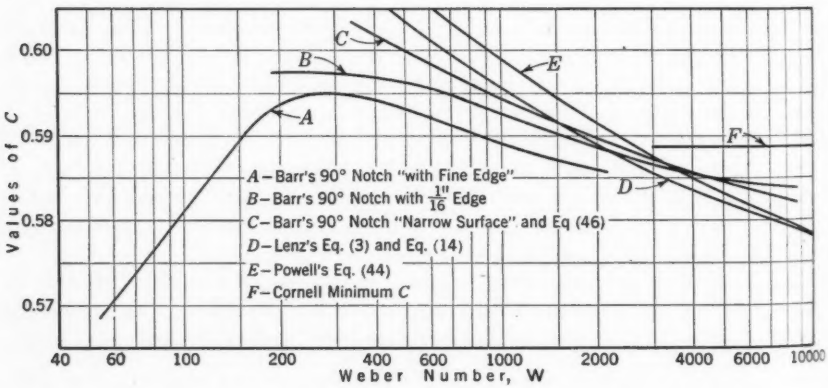


FIG. 20

unfortunately was plotted against the Reynolds number, a procedure which this paper shows to have been quite beside the point.) Curve *A* of Fig. 20 is the most striking, and it is interesting that in this semilogarithmic plotting it

[19] "A Study of Foam Formation in Aqueous Solutions," by Virgil Leland Hansley, an unpublished doctor's thesis, Ohio State Univ., Columbus, Ohio, 1928, p. 110.

[20] "Mechanics of Liquids," by Ralph W. Powell, Macmillan Co., 1940, Fig. 109, p. 199.

is perfectly straight from $\mathbf{W} = 54$ to $\mathbf{W} = 166$. The similarity between this curve and that of orifice coefficients[21] plotted against $\mathbf{R}$ is also worthy of note. The fact that curves $A$, $B$, and $C$ are so different may be partly due to the fact that the writer's assumption of the same temperature for the three cases introduces an appreciable error, but it is more probable that the slight difference in the three weirs was responsible. If so, before the V-notch weir can ever become an instrument for absolute measurement (dependable without calibration), the size of plate, thickness of edge, and angle of bevel must be standardized more definitely. (Barr's original data included a record of the temperature, as shown in the sample that was printed. Is it possible that these records are still in existence, so that $\mathbf{R}$ and $\mathbf{W}$ could even yet be found for each of his runs?)

For water at the temperature assumed, $\mathbf{R}$ and $\mathbf{W}$ were computed for a number of different heads, and from these $C$ was computed by Eqs. 3 and 14, and 44, and the results plotted in Fig. 20. As the author's edge was $\frac{1}{32}$ in., it would be expected that his results at this temperature would lie between curves $A$ and $B$, but nearer curve $B$. (Curve $B$ is the one from which Barr's values, plotted in Fig. 15, are taken.) To make the discrepancy no more than 1% would require changing the minimum $\mathbf{W}$ from 300 to about 600. As a matter of fact, any formula of the kind proposed by the author (or the writer) could not be expected to fit beyond the point of inflection of the curve, and for curve $B$ this seems to be at about $\mathbf{W} = 900$. For curve $C$, however, it is at least as low as the lowest $\mathbf{W}$ on the curve—that is, 340. This is the curve fitted by Mr. Strickland,[18] and changing his formula into one similar to Eq. 44 gives

$$C = 0.565 + \frac{0.435}{\mathbf{R}^{1/3}} + \frac{0.102}{\mathbf{W}^{1/4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (46)$$

This fits curve $C$ with a maximum discrepancy of about 0.04%. There are other pairs of numerical coefficients, of course, which would fit equally well—any that would reduce to

$$C = 0.565 + \frac{0.0157}{\sqrt{H}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

Undoubtedly similar formulas could be deduced to fit curves $A$ and $B$ up to the point of inflection with not much greater discrepancies.

That the author found, for very large values of $\mathbf{R}$ and $\mathbf{W}$, that $C$ becomes independent of either is not surprising in view of the analogous facts regarding the coefficients for orifices, nozzles, and pipes (and, the writer believes, for channels as well); but it is somewhat surprising that the limit is reached for both $\mathbf{R}$ and $\mathbf{W}$ at precisely the same value of $C$. Also, the fact that at high heads all of Barr's weirs gave values of $C$ less than 0.5886 would seem to indicate that this limiting value of $C$ depends on the weir thickness (as well as its sharpness and the smoothness and size of the upstream face).

Finally, physical chemists have discovered that certain solutions, such as 500 ppm of peptone in water or 100 ppm of saponin in water, have what is

[21] "Mechanics of Liquids," by Ralph W. Powell, Macmillan Co., 1940, p. 190.

called "surface viscosity."[22] This is quite different from ordinary bulk viscosity and seems to be due to the formation of a plastic surface film. It is not likely that the liquids used by the author had this property, but it would seem probable that, if a liquid had it, its flow over a notched weir would be affected thereby. Therefore, if experiments were to be made on a saponin and oleic acid solution as suggested herein, this matter should be investigated.

[22] "A Study of Foam Formation in Aqueous Solutions," by Virgil Leland Hansley, an unpublished doctor's thesis, Ohio State Univ., Columbus, Ohio, 1928, pp. 41–55.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

# UNUSUAL EVENTS AND THEIR RELATION TO FEDERAL WATER POLICIES

## Discussion

BY MESSRS. DAVID J. GUY, AND ALBERT S. FRY

DAVID J. GUY,[16] ESQ.[16a]—Without oversimplification, the author has isolated the germs of a revolution that is taking place in the policy of the United States with respect to water resources. The combination of circumstances cited—one physical, one economic, and one altruistic—is producing (in fact has already produced) a complete reorientation in the water-utilization program. No one can challenge the fact that the decade 1930–1940 was characterized by unusually large and destructive floods. The author's statement that similar climatic and hydrologic conditions have "probably" not been exceeded during the past century is accepted.

However, for a proper evaluation of this condition as one of the combination of circumstances shaping national policy, the reader requires a fuller historical background. The impact of unusual climatic and hydrologic conditions upon national life depends upon the character of national life at the time of their occurrence. To illustrate, science reveals that there is no sound—there are only sound waves—if there is no ear to intercept the sound waves. It is so in large degree with the occurrence of unusual floods. Population density, economic wealth, and encroachment of structures upon the natural flood plains and channels of rivers are determining factors in appraising flood effects and therefore the wisdom or unwisdom of any given flood-control policy.

When levee construction along the Mississippi River began at New Orleans, La., in 1717, small earthen dikes from 4 ft to 6 ft high were all that was necessary to afford protection from floods. At that time this appeared to be a wise flood policy, although future years proved that these same levees created a bottleneck that accentuated flood dangers; and, as other sections along the river above New Orleans built dikes, robbing the river of its flood plain, the race for

NOTE.—This paper by W. G. Hoyt, M. Am. Soc. C. E., was published in February, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1942, by Messrs. Edward H. Sargent, Dana M. Wood, James S. Sweet, and J. L. Campbell.

[16] Hydr. Engr., Natural Resources Dept., Chamber of Commerce of the United States, Washington, D. C.

[16a] Received by the Secretary June 26, 1942.

higher and higher levees became a flood-creating flood-preventing device, robbing Peter to pay Paul.

The passage of the Swamp-Land Act in 1850 established another "water-regulation policy" and that, from the backsight of today, had many unwise features. The author's paper deals with land and water, and it may also be pointed out that numerous recent land-policy discussions, appearing in print, have cast much unjust criticism at the early land-settlement policies of this country, particularly the Homestead Act. Fairness requires that the early policy makers in the United States, dealing with land and water, be credited with enacting laws and putting into effect policies deemed wisest by them in the light of the problems of their time. Even today there are lawsuits in California because of efforts to prevent river floods that overflow and replenish the ground water supply of the lower plains area.

This subject of national water policy calls for a more extensive plotting of points and circumstances before anyone can indicate a trend line with confidence. An old farmer once said he never had any trouble getting two fence posts in line, but when he had three he ran into difficulty.

Land and water policies should be founded upon scientific and engineering inquiry. Unfortunately, "The impact of an economic crisis," to which the author refers (see "Introduction"), has been more persuasive with the legislative policy makers. It is questionable whether a federal water policy shaped by the ebb and flow of unemployment can be judged engineeringly sound. It is also questionable whether the use of soil and water conservation as a Constitutional peg for economic adjustments in agriculture should be admitted by the engineering profession as a factor in shaping national water policies. This is not to say that the engineer should avoid all consideration of matters beyond those that are technical and immediately "profitable." The danger is that any water policy may be literally pulled to pieces for no other purpose than to make legal and Constitutional pegs on which to hang a variety of new social and political philosophies.

The current water policy is enmeshed in the "shelf of projects" that has been built up, awaiting the "go" sign when jobs are needed after the war ends. Many of these projects are for the purpose of conserving and using water resources. As experience gained in the past decade has shown, economic soundness does not necessarily control the order of development when the objectives sought lie outside the bounds of an integrated water policy.

Emergency relief through make-work projects was made readily acceptable by the generous use of the federal treasury. Appropriations through the various emergency relief acts of the period 1935–1942 exceed $15,000,000,000. Never before has the United States accepted the lure of debt and borrowing as witnessed in the decade 1930–1940, reviewed in the author's paper on water policy.

This decade witnessed generous use of federal loans and grants to states and municipalities for the development of water resources. The long-established policy of the Army Engineer Corps that required local contributions toward the cost of river-improvement structures was seriously threatened by the 1936 Flood Control Act, and was practically eliminated by the 1938 Flood Control

Act. The complete elimination of local contributions tends to make flood control a "pork-barrel" proposition. Heretofore, the willingness to share the cost was the best index of a local need of flood control. The principle, rather than the amount, of local contribution is important. It may be, as the author states, that the great losses during the 1930–1940 decade from floods and droughts "made a deeper impression on policy makers, than ever before" (see "Introduction"), but public acceptance was achieved largely because of the federal Santa Claus. Here is one obvious reason "for the rapid transition of a wide popular movement or belief into an active action program" of which the author speaks (see "Introduction"). Here is a kind of inverted "dollar diplomacy." Dollar policy making carried with it certain objectives and created certain obligations. This has long been the situation growing out of various forms of federal aid—to wit, the beneficiary surrenders something, usually a measure of local control.

It is no surprise, therefore, that many new federal agencies were created—some by Congress, some by Executive Order—to exercise the rights and powers yielded by states and communities. Widespread federal planning came into vogue. It now embraces every possible land and water use, and extends from the smallest mountain stream to sea level. The Nation is now in a period of intensive federal water-policy making accompanied by a rapid decline in state and local control.

The extension of federal water policy does not stop at planning. It is being projected into the fields of operation and management, prompted, at times, by political rather than engineering considerations. Again the "Constitutional peg" comes into play. The licensing policy of the Federal Water Power Act established in 1920 has been abandoned for direct federal construction of water-power plants and federal marketing of the energy generated. This latter, a marketing phase, is still in a much muddled state. The political desire for public power has been of equal or greater influence in shaping present water-power policies than any circumstance of the "1930–1940 flood situation" to which the author refers.

"Multiple-use projects" has become a slogan which has political as well as engineering significance. This multiplicity of uses has greatly extended the field of federal participation and has served to divest states of much of their control of water resources. "Multiple use" presumes an ability on the part of planners to make fine economic adjustments in water-use values that, from an engineering approach, can be little more than wishful thinking. Take, for example, the question of allocating cost between flood-control and water-power benefits. Army engineers used to measure allowable flood-control expenditures by capitalizing flood losses. The best that can be said for this practice is that it served to indicate the relative importance of the regions damaged. The need for flood protection, and therefore the amount of expenditures, is determined by the measure and extent of the fear the endangered people feel, due to the recurrence of floods. It is this fear that generates the pressure on the members of Congress, state legislators, and the "city fathers" to appropriate the money for flood control. The methods of control are for the engineers to determine.

With water power, the former practice was to measure its economic value by comparison with the cost of steam power. The current policy seems to be to allocate to power only such a share of the capital investment as will make government power "cheap" in comparison with prevailing prices of private utility companies. This might be an acceptable policy if its object were for no other purpose than to assure the disposal of the government's power by-product at multiple-purpose projects. The trend of the policy, however, is to undertake federal dictation of retail rates for electric service, a long recognized state function.

The "multiple-use" policy has not developed into a true water-conservation policy such as was envisioned 40 years ago by such men as George Maxwell and the late Frederick H. Newell, M. Am. Soc. C. E. The nearest approach to water-conservation policy in recent years is found in the program for upstream development furthered by the Wheeler-Case Act and the Taylor Grazing Act. This departure in water policy, so well presented by the author, stands in marked contrast to the earlier flood-control and navigation-improvement policy which was so rigidly confined to the lower reaches of major streams.

The stimulus to fact finding and hydrologic research, which has come about not only through the demand for more and better data but also through the generous allocation or appropriation of funds to these ends, is a valuable part of the new policy from the engineering standpoint. It is to be hoped that the politicians and planners eventually will be guided more and more by this research than by politics and social philosophy.

The writer concurs with the author's concluding expectation that only future experience can show the soundness or unsoundness of the prevailing water policy and, therefore, the need for modifications. Only by gearing the planning to a practical, and, therefore, immediate future can the needs of today and the probable needs of tomorrow be balanced with the greatest measure of justice and practicability. Such planning, to be of maximum value, requires constant adjustment in the light of social and economic changes and, one may well add, in the light of what the author describes as abnormalities in climatic and hydrologic conditions. In its very best sense politics must be a matter of issues; and national water policies, in regard to both practice and planning, should be based upon economic and engineering facts and should be free of such issues.

ALBERT S. FRY,[17] M. AM. SOC. C. E.[17a]—An excellent summarization of the developments and interrelations of federal water policies during the decade between 1930 and 1940 is presented by the author. The paper is particularly noteworthy because it presents a vast amount of material in condensed form of a readable length.

Perhaps somewhat more significance should be placed on the great flood of 1927 in the Mississippi Valley as influencing a change in federal water policies, and the revolution in federal water policies might well be dated from that time rather than 1930. It was the great flood of 1927 that really demonstrated that

[17] Head Engr., Hydr. Data Div., TVA, Knoxville, Tenn.

[17a] Received by the Secretary August 3, 1942.

flood control is a national problem far too big for local interests even in good times. The federal legislation that resulted in 1928, following this flood, recognized flood control for the first time as a national responsibility. Although levees had been built for many years previously along the Mississippi River, this work had been done under the guise of navigation rather than flood control. The studies of the Army engineers, embodied in the Jadwin plan, and later developments showed that the ultimate flood protection of the Mississippi Valley required something more than levees, and a comprehensive reservoir plan was developed for the Ohio River Basin. This plan was a stepping stone to the flood control acts of 1936 and 1938.

The creation of the TVA in 1933 marked not only the inception of a multiple-use project by the federal government, as the author states, but it also created for the first time a complete watershed development project with all phases of that development under one federal agency. As the author has indicated, other federal legislation has provided for cooperation between various governmental agencies, such as the Corps of Engineers and the Department of Agriculture, but only in the TVA has the complete water control of a large drainage basin, including water control both on the land and in the streams, been authorized and established under one agency. The success of this method of handling the water problems of a region has resulted in legislation being introduced into Congress to create other similar authorities.

This completely unified type of organization has facilitated the planning of water-control projects, utilizing the contributions of various sciences such as hydrology, forestry, and agriculture, and appears to have distinct advantages in the development of a complete water-control plan for a region over the more cumbersome procedure of planning by two or more separate governmental agencies, each jealous of its own particular field in the culmination of the final plan. Integral with the TVA type of organization is a recognition of the basic hydrologic information needed for complete water-control planning from all standpoints and maximum utilization of the data by the several sciences involved. Illustrative of this may be mentioned the pioneer work of the TVA in the extension of the network of rainfall stations throughout the Tennessee Valley, particularly those on higher altitudes for which no accurate information as to rainfall amounts and intensities previously existed.

A most beneficial result of recent federal water policies has been the development of closer cooperation between various government agencies concerned with water projects and the recognition of the interrelationships of the interests of these several agencies in specific problems. The TVA, for example, although organized to handle completely the water problems of its region has, from its beginning, recognized the value, and pursued a policy, of close cooperation with existing governmental agencies to avoid duplication of effort. Stream-gaging work for the TVA is done by the U. S. Geological Survey with which there has been a close and most satisfactory cooperation throughout the life of the Authority. This relationship has resulted in an increase in the scope and importance of the work of the Survey in the Tennessee Valley region. Where formerly stream-gaging stations were operated in the Tennessee Valley, and the results published years later in *Water-Supply Papers*, immediate problems of

planning and construction of TVA projects have resulted in stream-flow information being obtained from an increased number of stations and kept up to date for the day-to-day needs of the Authority. The making of stream-flow measurements below a large dam on the Tennessee River in the morning to be used in connection with operations in the afternoon is quite different from collecting records to be worked up and published years later.

Another example of a cooperative relationship is that between the Weather Bureau and the TVA with respect to weather forecasting, which is utilized in the operation of the many reservoirs in the Tennessee Valley. This unique arrangement provides the TVA with weather forecasts twice daily or oftener and the TVA in turn forecasts river stages throughout the Tennessee Valley, the information on both weather and rivers being given to the public in the form of a cooperative daily bulletin. It was recognized by Weather Bureau officials that, although the Bureau is required by law to forecast river stages throughout the United States, since the TVA necessarily had to do much more detailed forecasting in connection with its reservoir operations, it would be inefficient for the Weather Bureau to duplicate this work.

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DISCUSSIONS

# MECHANICS OF CREEP FOR STRUCTURAL ANALYSIS

## Discussion

BY MESSRS. A. HRENNIKOFF, C. A. P. TURNER, AND RALPH W. STEWART

A. HRENNIKOFF,[15] ASSOC. M. AM. SOC. C. E.[15a]—The writer wishes to call attention to two mistakes made in the derivation of the theory of creep as presented in the paper of Professor Marin.

*Creep Law.*—For the basis of his theory the author (see Eq. 7*a*) selects the following relationship between the unit stress $s$ and the unit strain $e$: $e = s^n (B + a\, t^m)$, in which the exponent $n$ depends on the properties of the material and may assume any integral or fractional positive value above unity.

In order that a stress-strain relationship could be used to advantage in the derivation of a theory of flexure it should be valid both for the positive and the negative values of $e$ and $s$—that is, both for the state of tension and the state of compression, a requirement which, of course, is satisfied by the familiar formula that expresses Hooke's. This condition, however, is true with regard to Eq. 7*a* only when the exponent $n$ signifies an odd integer. On the other hand, when $n$ assumes the value of an even integer or of a fraction, this equation under the conditions of negative stress becomes inconsistent, since the left-hand side of it is then negative, and the right-hand side positive when $n$ is an even integer and imaginary when $n$ is a fraction.

Eq. 7*a* would be correct, of course, from a mathematical point of view if $e$ and $s$ were used in the sense of the numerical values of stress and strain. This, however, was not the intention of the author, and with this interpretation Eq. 7*a* would scarcely be suitable for the purpose intended.

The error discussed herein leads to inconsistencies in further derivations. Thus, the basic differential equation (Eq. 18) relating the value of the bending moment in the beam with the value of the curvature of the elastic line $\frac{d^2y}{dx^2}$ is incorrect when $n$ is an even integer or a fraction, since according to this equation

NOTE.—This paper by Joseph Marin, Assoc. M. Am. Soc. C. E., was published in May, 1942, *Proceedings*.

[15] Asst. Prof., Civ. Eng., Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

[15a] Received by the Secretary June 4, 1942.

a negative bending moment bends the beam in the same sense as a positive bending moment when $n = 2, 4$, etc., or produces an imaginary curvature when $n$ is a fraction.

The same error is naturally carried into the solutions of the statically indeterminate structures, discussed later in the paper. Take, for instance, Example 1 depicted in Fig. 6. The symbols $m$ and $M_B$ in this figure denote the numerical values of the end moments, since the arrows representing these quantities are shown in their true directions. The relationship between these moments is expressed by Eq. 26*b*, an inspection of which shows that, should $n$ be taken as a positive even integer, no positive values of $m$ and $M_B$ may be found to satisfy this equation—that is, to satisfy the conditions of the problem.

The author has been unlucky in selecting for the illustration of his theory in Table 3 the numerical values of $n = 1, 3, 5$, and 7—that is, those specific values for which his theory still holds. Had he used any other values of $n$ the fallacy of his theory would have manifested itself in the absurdity of the numerical results.

*Moment-Distribution Method.*—The peculiarity of the stress-strain law used by the author makes superposition of deflections in beams caused by different loadings inadmissible. As an illustration of this principle, imagine a simply supported beam of a rectangular cross section loaded with a concentrated couple $M_A$ on the end $A$ and free from any other applied loads. The couple $M_A$ causes rotation of the end $A$ through an angle $\theta_A$. Imagine now a new couple of the same magnitude $M_A$ applied to the same end in the same direction. The second couple doubles the original moments caused by the first couple, as follows directly from statics, and doubles the original stresses anywhere in the beam, as may be concluded from inspection of Eq. 17. The strains, however, are more than doubled in view of the nature of the stress-strain law assumed, because the beam fibers are working now on the flatter parts of the stress-strain curve than they were before. Larger strains result in larger deformations, and the additional angle of rotation of the end $A$, produced by the second couple $M_A$ alone, is larger, therefore, than the original angle $\theta_A$; that is, the presence of the first state of stress in the beam has made the beam more flexible on the additional loading.

This behavior of the beam has an important bearing on the validity of the moment-distribution procedure, and it makes evident the fact that in the beam just considered the distribution factor on the second loading is smaller than on the first loading, provided that the other members attached to the same joint were still unstressed before the second loading. This dependence of the distribution factor (and of the carry-over factor as well) on the state of stress already existing in the beam may be demonstrated also for the fixed-ended beam by going through the necessary algebraic derivation. The author did not detect this dependence because his beam had been unstressed previous to the application of the end rotation. Dependence of the value of distribution factor on the state of stress in the beam makes the version of the distribution procedure presented by the author erroneous.

A modified moment-distribution procedure allowing for the state of stress existing in the beam is conceivable theoretically but scarcely possible in

practice. It would be too laborious, since each step in the course of distribution would require a new set of values of the factors in each span of the structure.

The conclusion reached concerning the applicability of the method of moment distribution should not be surprising if one realizes that this method is based on the principle of superposition.

Summarizing the discussion it may be stated that the theory of creep as presented by Professor Marin is valid, from the mathematical point of view, only in the following two cases:

1. When the exponent $n$ is an odd integer; and
2. When the signs of the bending moments are known beforehand in all parts of the structure. The value of $n$ under these conditions may be arbitrary, but some modification in the author's formulas will be necessary if the sign of the moment is not the same everywhere in the structure.

The moment-distribution procedure is inadmissible even in cases when the theory holds.

It must also be kept in mind that the theory presupposes constant loads and seems to be inapplicable to simultaneous action of loads of different duration.

C. A. P. TURNER,[16] M. AM. SOC. C. E.[16a]—Professor Marin is to be congratulated for bringing into question the validity of treating Hooke's hypothesis as a law of nature. It may be doubted whether Hooke was first to present the suggestion that strain is approximately proportional to stress, since it appears in the Mariotte-Leibnitz theory of flexure following the publication of Galilei's problem.

At that early date, caloric or heat, light, electricity, and magnetism were regarded as imponderable chemical elements. This viewpoint (that of complete ignorance of intermolecular forces which oppose elastic change in form) warranted the adoption of the infinitesimal point hypothesis of Boscovich, which embodied a complete disregard of the temperature change incident to the application of the load.[17]

Now this temperature change which takes place upon the application of the load furnishes a measure of the magnitude of the load, as described by the writer in 1902.[18] The first practical recognition appears in the work of testing at the rolling mill where, by varying the speed of the machine in testing, the operator could satisfy engineers who wanted soft steel or others who wanted medium steel from the same metal with a different speed in testing. Thus, from the same bar, it was possible to develop, for soft steel, a 40,000-lb to 41,000-lb yield point with an ultimate strength of 58,500 to 60,000 lb per sq in., thus satisfying the college-trained engineer who wanted soft steel; and again,

[16] Cons. Engr., Columbus, Ohio.

[16a] Received by the Secretary June 25, 1942.

[17] "An Elementary Treatise on the Mathematical Theory of Perfectly Elastic Solids, with a Short Account of Viscous Fluids," by William J. Ibbetson, Macmillan and Co., London, England, and New York, N. Y., 1887.

[18] *Transactions*, Am. Soc. C. E., Vol. XLVIII, August, 1902, p. 140.

by increasing the speed in pulling the bar, a yield point of 44,000 to 45,000 lb per sq in., and an ultimate strength of 60,000 to 62,000 lb per sq in., could be developed, thus satisfying the engineer who wanted medium steel!

The energy of this temperature change upon application of the load depends on the final state and is independent of intermediate states as opposed to the external work, which is measured kinetically by the mean force times the deformation (by "kinetic," the initial equilibrium of the external and internal forces is meant).

The shift of internal energy resisting the external force is kinetically double the external work, and from this relation the physical properties of the metal may be coordinated.[19] As the elastic strain thus involves not only elongation but temperature change as well, it is apparent that the kinetic or instantaneous balance of equilibrium of external force and internal resistance will shift to the static balance as the thermo-elastic temperature change is dissipated by radiation. It further appears that to the extent the specimen is homogeneous the temperature change will be proportional to the applied force and, when this thermal limit of elasticity is exceeded, a range of imperfect elasticity is encountered before the development of the yield point of the metal. As this range of imperfect elasticity is developed, the cooling effect of tension is reduced until a radical heating effect is developed, with flow of the metal.

The shift from kinetic to static equilibrium was discovered by Kelvin, who attributed the effect to viscosity instead of the radiation of temperature change incident to the application of the load. Kinetically, the internal energy shift is double the external work, but this kinetic external work is but five sixths of the static, as may be demonstrated in many ways by different formulas. Hooke's hypothesis is in no sense a law of nature; it is inexact unless the time element is taken into consideration.

In his work on the manufacture of steel, H. H. Campbell[20] found that by applying the load on the structural specimen at the rate of 0.1 in. per min, a yield point of 39,000 lb per sq in. was developed, whereas with the fast speed of the machine a yield point of 45,000 lb per sq in. was developed.

To cover the phenomena of viscosity within the thermal limit of elasticity, one may state that, if the load remains in place at a constant intensity until the temperature change on its application has been dissipated by radiation, the static elongation is proportional to the load applied and there is no creep or increased elongation. The situation is altered by residual strain from cold bending or other harsh treatment.

Residual strain presents a kind of elasticity of its own, best illustrated by cold bending some $\frac{3}{8}$-in. square rods in a half circle with a 3-in. radius; then placing the horseshoe type specimen in a vise and sawing kerfs $\frac{1}{8}$ in. deep and $\frac{1}{2}$ in. apart, the distance between the ends of the bent piece will be decreased when taken from the vise. Reversing the process and sawing the kerfs on the outside of the bend, the ends will again spring in or will present a sharper curvature; then sawing the kerfs on both sides, the horseshoe will open out

[19] "Elasticity, Structure, Strength and Chemical Action of Materials," by C. A. P. Turner, Pt. III.

[20] "Manufacture and Properties of Structural Steel," by H. H. Campbell, 1896.

and decrease its curvature. This change in form is caused by upsetting the internal equilibrium between the internal residual and elastic areas of stress.[21]

When extensometer measurements of stresses in concrete floors were a fad with student investigators, it often happened that they measured, according to Hooke's law, residual changes in a form which they misinterpreted as elastic strains one and one-half to one and three-quarters times as great as the yield point of the metal they were investigating. (Measurements across cold bends or kinks in small rods give rise to erratic results.) This kind of residual creep has never received the attention it deserves.

Prior to Wohler's[22] experiments on repeated stress, the elastic limit of iron and steel was considered little less than the yield point of the metal. With range of stress from tension to compression exceeding $\frac{5}{8}$ of the primitive yield point, failure occurred under a sufficiently large number of repetitions.

Instead of recognizing a range of imperfect elasticity beyond the natural elastic limit, the metaphysical range of stress formulas of Launhart and Weyrauch was developed based upon the assumption of fatigue within this natural limit where none was developed by experiment.

With the passage of time these ranges of stress formulas have been discarded.

The writer's experiments on thermo-elastic measurement of stress place the thermal limit of elasticity as approximately five eighths of the yield point, in harmony with the work of Wohler. Creep or increased deformation will occur with time under stresses beyond the thermal limit of elasticity because, when this limit is exceeded, work has been done in displacing the molecular arrangement.

Recalescence points in melting or cooling of metals are recognized as points where the molecular formation undergoes modification. For example, soft ingot metal of the cubic system presents molecular kernels of eight atomic spheres. Upon rolling, this structure is broken down, and body-centered, nine-atom kernels appear. Again, drawing into fine wire, the body-centered formation changes to a face-centered formation, and the increase in strength under the law of attraction is as the square of the number of atoms in the kernel.

*High-Temperature Creep.*—At higher temperatures, a shift of molecular structure may take place in a manner similar to that occurring at the recalescent points of the metal, developing the so-called phenomena of creep at high temperature.

No natural phenomenon has been so much misrepresented by metaphysical hypotheses as that of elastic reaction. Volume, rather than elongation, is the correct measure of the effect of an axial force. Tension increases volume and compression reduces volume, and, although the same force causes equal volumetric changes whether from tension or compression, the temperature changes differ less in magnitude for tension and more for compression. In accordance with the law of volume, increase of volume is endothermic and decrease of volume is exothermic. This fundamental physical law destroys

[21] "Elasticity, Structure, Strength and Chemical Action of Materials," by C. A. P. Turner, Pt. I, p. 55.
[22] *London Engineering*, Vol. XI, 1871.

the basis of Lorentz and Einstein's calculation[23] of the interconvertibility of energy and mass and proves the irrationality of that characteristic energy of matter developed mathematically from a literal expression of Hooke's hypothesis. In a conservative system, the distribution of energy depends on the relation of masses and is irrespective of the individual unit, as demonstrated in the philosophical treatment of *The Encyclopædia Britannica*.[24]

The fantastic idea that one-quarter gram of matter has an intra-atomic energy equivalent to that developed by the engineer in burning 5,000 tons of coal is a metaphysical dream.[25] Not only has the elastic action of solids been confused in theory, but, in elastic resistance of gases to compression, physicists have invented a perfectly imaginary ideal gas in which the molecules have no volume and occupy no space. Because all gases have been liquefied and liquids are almost incompressible, one needs only to deduct the volume of the liquefied gas from the gross volume to find the net volume capable of compression in accordance with the laws of Charles, Boyle, and Gay-Lussac.

Professor Marin's creep theory for beams embodies the assumption that a plane section before bending remains plane after bending. Among the infinite number of sections in the length of the beam, one only in a simple beam remains plane after bending. That section divides the moment area in equal parts. As applied to the restrained and continuous beam, notwithstanding the creep correction of Professor Marin, the maximum tensile stresses at midspan and those over the support will be calculated anywhere from 15% to 30% greater than actual and the maximum compressive stresses will be approximately correct. The deflections will be much less than those computed in the restrained and continuous beams and greater than those computed by the ordinary formula for the simple beam.

The most curious inconsistency of the "Ut tensio sic vis" (extension proportional to force) lies in the fact that, under longitudinal compression, lateral extension occurs without any force or component of the applied force to account for it. Some have endeavored to explain this by a misconception of shearing stress; but the shears, as in beam action, can be developed only by principal stresses of opposite signs, and one of these principal stresses has no external force in its direction of action to account for its development.

The elastic temperature creep (that due to radiation of temperature change on application of the load) discloses the irrationality of the dynamic stress theory because the kinetic external work is but five sixths of the static; doubling it would be one and two thirds. Because the yield point before radiation is six fifths that after radiation is complete, the general effect of the suddenly applied load in developing the yield-point value is but one and one-third times the gradually applied static load, instead of twice the load, as taught in handbooks and textbooks of today.

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[23] *Webster's New International Dictionary* (unabridged), 2d Ed., 1934, p. 847—$m = \frac{E}{c^2}$; "Problems of Modern Physics," by H. A. Lorentz, Ginn and Co., Boston, Mass., and New York, N. Y., 1927; "World of Atoms," by A. E. Haas and H. S. Uhler, Van Nostrand, New York, N. Y., 1937, pp. 112–113; and "Structure of Matter," by Cranston, p. 9.

[24] *The Encyclopædia Britannica*, 9th Ed., p. 207; also "Elasticity, Structure, Strength, and Chemical Action of Materials," by C. A. P. Turner, Pt. III, pp. 49, 51, and 141.

[25] "A Comprehensive Treatise on Inorganic and Theoretical Chemistry," by Joseph W. Mellor (Longmans, Green), London, England, 1922.

Initial tests of newly cured reinforced-concrete beams disclose that steel stress times its lever under moderate loads is customarily 20% to 30% less than the applied moment. This inequality creeps out of existence under repeated loading. Textbook authors point to it as evidence of the operation of tensile resistance of concrete; but, by turning the concrete beam bottom side up, evidence of its insignificant value of concrete tension in flexure appears, requiring a more logical explanation.

As the concrete matrix crystallizes, it expands, inducing tensile stresses in the steel of a magnitude of 4,000 to 6,000 lb per sq in. The reaction to these steel stresses induces a state of residual compression in the concrete about the metal and, because the reaction to residual strain is antipodal to its elastic counterpart as shown by the cold-bent horseshoe bars, the resisting moment is that of steel stress times its lever plus residual compression in the concrete times the same arm, with the joint effect equilibrating the external moment.

Some deformed reinforcing bars indicate their top and bottom surface under their final pass through the rolls. If, on an 18-in. specimen, saw kerfs are made in the vise at ½-in. intervals, the bar will curve up slightly because the residual strain from rolling has been thrown out of balance by the reduction of sectional area. As in the example of the initially cured concrete beam, such residual strains are dissipated by repeated loading. After years of painstaking investigation, Bauschinger[26] concluded that sound metal, rather than being fatigued or brought nearer to rupture by alternating stresses from tension to compression within the natural elastic limits, was improved in elasticity, ductility, and tenacity.

An interesting example of apparent creep or downward motion of the neutral plane of reinforced-concrete beams was developed by the National Bureau of Standards[27] under the direction of the late Richard L. Humphrey, M. Am. Soc. C. E., in connection with the joint committee of the Society. In locating the position of the neutral plane, Mr. Humphrey and his committee assumed that this locus would be found where the longitudinal compression passed through zero, as in homogeneous beams, but that this locus crept downward as the load increased. In the reinforced-concrete beam, the shears between the tension of the steel and compression in the concrete are equilibrated at the steel instead of at the locus of the centers of rotation.

In the stress mechanism of a homogeneous beam, the lines of greatest compression above the neutral plane pair with the lines of greatest tension below it and intersect the neutral plane at 45°. In the concrete beam, because the shears are equilibrated at the bottom about the steel, the states of right shearing stress as centers of rotation are twisted 22.5° from their position in the homogeneous beams. This causes the lines of greatest compression to pair with the lines of greatest horizontal shear, intersecting the neutral plane at 22.5° instead of 45° so that the neutral plane is compressed longitudinally in proportion to the applied load, causing the locus of zero compression to creep downward as the load is increased. Elastic deflection of a concrete

[26] "Concrete-Steel Construction," by H. T. Eddy and C. A. P. Turner, Minneapolis, Minn., p. 109.
[27] *Technological Bulletin No. 2*, National Bureau of Standards' experiments on concrete beams.

beam, therefore, increases more rapidly than the load because of this compression and shortening of the neutral plane.

Failure to grasp this relation has led to the unscientific conception of parabolic stress distribution in the compression zone and general confusion and mystery regarding the co-action of metal and concrete, limiting the conception of the field of investigation to that of concrete ribs functioning as steel beams would do.

No progressive achievement resulted from the deliberations of the Humphrey committee because they failed to grasp the idea of monolithic action in economic construction and the nature and distribution not only of shearing stresses in beams but of twisting stresses in plates. The stress mechanism of both homogeneous and reinforced-concrete beams and plates has been developed by the writer.[28]

The theory of elastic reaction as treated under Hooke's hypothesis is inconsistent with the statics of abstract forces because lateral extension occurs under longitudinal force with no component of that force to account for its development. In thermo-elastic theory this inconsistency disappears and Young's modulus may be coordinated with specific heat; coefficient of expansion; unit weight; atomic weight and number; strength; and melting and boiling temperature through $J$ (the mechanical equivalent), $g$ (the gravitational constant), the gas-volume coefficient (constant temperature), or the gas-pressure coefficient (constant volume).

Thermo-elastic theory, treating actual rather than imaginary substances, embodies nothing more complicated than common school arithmetic, which should render it popular with the poor student who has labored with the tedious differential equations of mathematical theory, unrelated to the physical properties of actual substances.

The college textbook theory of the nature of tension and compression embodies the spheres of exclusion developed in 1650, and the foundation of mathematical elasticity is the infinitesimal point hypothesis of Boscovich of that same modernistic date and generation.

RALPH W. STEWART,[29] M. AM. SOC. C. E.[29a]—The gradual increase in permanent deformation of a highly stressed structure, which the author calls "creep," has been given little attention in civil engineering literature. In reading this paper the writer was impressed with the fact that it contained valuable information on the subject.

In the writer's early practice he noticed that the 20-yr-old wooden joists in certain heavily traveled bridges had acquired a pronounced sag due to creep. For long beams continuous over a number of supports, the effect of creep obviously would tend to equalize the positive and negative moments after the manner of the plastic flow in an overstressed structure. In fact, the distinction between creep and plastic flow is narrow and seems to depend on the time element.

[28] "Elasticity, Structure, Strength and Chemical Action of Materials," by C. A. P. Turner, Pt. III, plates VIII and IX.

[29] Engr. of Bridge and Structural Design, City of Los Angeles, Los Angeles, Calif.

[29a] Received by the Secretary July 13, 1942.

In designing machinery there is evidently justification for including the effect of creep in the stress analysis, but in designing continuous bridges and building frames the neglect of shear deformations may produce greater inaccuracy than the neglect of creep.

One difficulty in including the effect of creep in a complicated structure is that at some points the stresses would be sufficient to cause creep but that at other locations (except on rare occasions) the stresses might be too light to cause creep. This would make the stresses due to creep discontinuous, whereas the elastic stresses would be continuous.

In two places the author indicates conceptions and conclusions that the writer considers to be erroneous—namely, the reference to end-moment distribution as "basic" in character and "fundamental" in nature, and the statement that it should be used for computing moments which include the effect of creep.

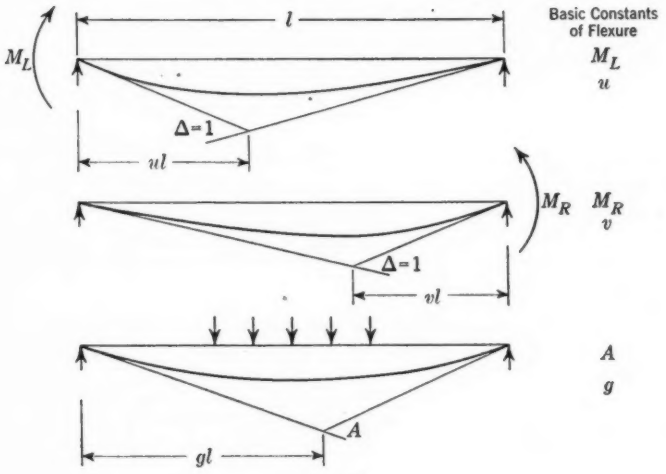


FIG. 13

The basic constants of beam flexure for any beam are illustrated by Fig. 13 which shows the elastic curves, their tangents, and their long chords for the three different conditions that yield the basic constants of flexure. The illustrations can be considered as depicting railway curves or any geometrical curves, and the properties may be considered as geometrical rather than mechanical. The basic constants of flexure read from these figures are shown in Fig. 13. They are the best and the minimum number of constants necessary to determine the geometry of the triangles which circumscribe the elastic curves.

The end-moment distribution constants are not basic but are "derived" constants. For tapering members or other special conditions they become complex.

The writer would define a solution of basic character as one which uses the basic constants of flexure and which does not use any form of converging approximations. Such a solution is shown in Figs. 14 and 15. It involves fewer

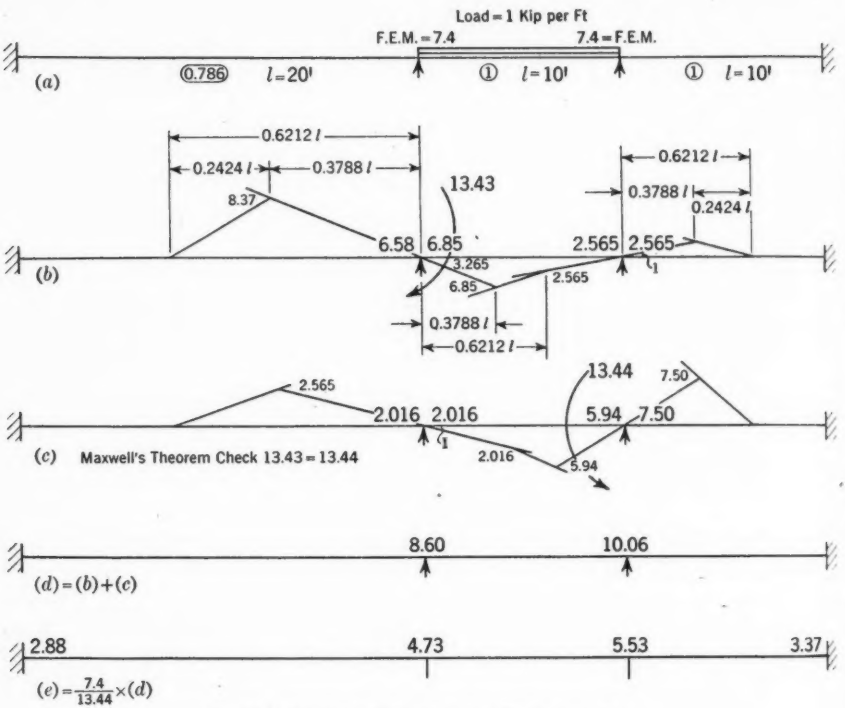


FIG. 14.—ANALYSIS INCLUDING THE EFFECT OF CREEP
(F.E.M. = Fixed-End Moment)

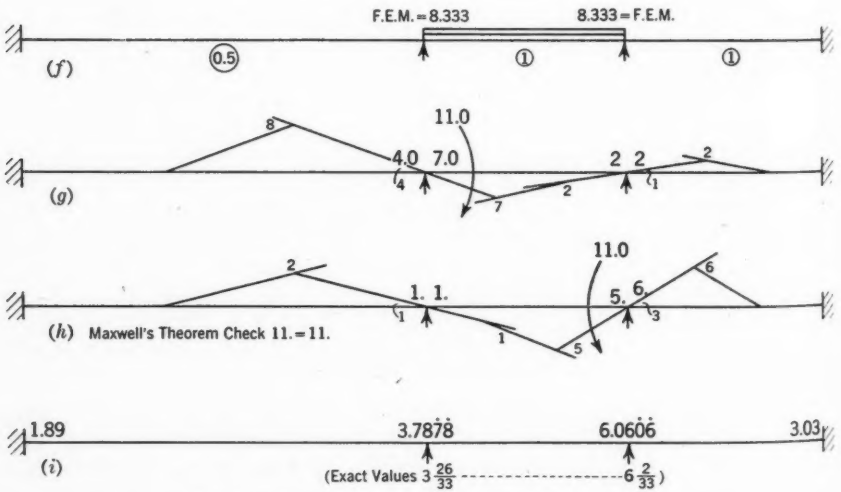


FIG. 15.—COMPARATIVE ANALYSIS FOR THE ELASTIC CASE
(F.E.M. = Fixed-End Moment)

arithmetical operations than the author's solution. The arithmetic is simpler, in that there are no long columns of figures to add, and its result is more accurate than the author's. It will compute moments to any predetermined degree of accuracy in one operation. It does not involve the use of simultaneous equations. Except for the last operation, it has an automatic check by Maxwell's theorem of reciprocal deflections. The writer therefore contends that the author's statements regarding methods of computation are incorrect and that they tend to perpetuate a now common practice of failing to select the most effective method for analyzing a large class of structures.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

# EARTH PRESSURE AND SHEARING RESISTANCE OF PLASTIC CLAY

## A SYMPOSIUM

### Discussion

BY MESSRS. RALPH H. BURKE, L. G. LENHARDT, GEORGE E. SHAFER, AND M. E. CHAMBERLAIN

RALPH H. BURKE,[21] M. AM. SOC. C. E.[21a]—The study of soil mechanics gives promise of effecting the next major advance in the design and construction of underground structures. The past three or four decades have witnessed great improvements in the manufacture of cement and steel, and to a lesser extent a standardization in the choice of aggregates, so that uniformity in the ordinary construction materials may now be assured. The technique of concrete proportioning has become fully recognized so that, with proper engineering organization, the failure of structures due to careless or ignorant workmanship is eliminated. The analysis of stresses, even though many underground structures are indeterminate, has been the subject of extensive research and has become rationalized to such an extent that full dependence may be placed on the structures to withstand their assumed loads. Design assumptions based upon anticipated earth pressures, however, still are irrational in the extreme and contain about as many unknowns as must have confronted the builders of the Egyptian pyramids.

The results of soil studies on the Chicago subway, as outlined in Mr. Peck's paper, are a valuable contribution to the engineering profession in a field that is still relatively unexplored. They indicate that, at a moderate expenditure, soil samples carefully taken and properly analyzed in the laboratory can be safely relied upon to disclose the physical characteristics of the soil. The field measurements have demonstrated that upon these test results reasonably sure assumptions as to the value of the lateral pressures against open cuts can be predicated. The results also indicate that, in tunneling through a wide variety of Chicago clays, with water contents ranging from 15% to 60%, the structures

NOTE.—This Symposium was published in June, 1942, *Proceedings*.

[21] Deputy Coordinator, Protective Service, Office of Civilian Defense, Chicago, Ill.

[21a] Received by the Secretary June 22, 1942.

tend to deflect to the shape that will produce horizontal pressures approaching the full vertical loads. In circular tubes that are sufficiently flexible, for example, it follows that the bending moments so commonly assumed to exist are not to be found in the finished structure. The results further indicate that, in an open cut, by the proper spacing of wales and bracing and by pre-stressing the struts to prevent deflections under load, the adjacent ground movements or settlement can be minimized.

These results go further than theory. They are of practical value. They mean dollars saved in the elimination of wasteful design. They mean dollars saved in the prevention of damage from faulty construction.

The design assumptions adopted for the Chicago subway were in accordance with widely accepted practise. Although it was felt that they were ultra-conservative, they were fully justified by precedent. In fact, in the absence of exact data at the time the work was started, any departure from the current practise would have been unwarranted.

The method most commonly used is to assume that the soil is a perfect fluid having an arbitrary density. This density is usually the result of majority opinion and has no rational background. To the credit of the engineering profession, it is usually assumed so great that underground structures do not fail when so designed. As a further factor of safety, instead of adopting only one value for density, stresses are analyzed on the basis of horizontal pressures, equal, respectively, to $\frac{1}{3}$ and to $\frac{2}{3}$ of vertical pressures, and the structures in each instance are made adequate to meet the most severe of these conditions. It is apparent that these assumptions disregard cohesion and other soil properties and are independent of the depth of the work. The experimental results on the Chicago subway indicate that both the soil properties and the depth are factors that affect pressures. In general, the fluid theory may be said to be irrational and in most instances will lead to uneconomical design.

The chief inherent fault of the fluid hypothesis is that the total lateral pressure is thus assumed to increase as the square of the depth, and to have a value modified somewhat by whatever angle of internal friction is assumed. In fact, however, the test results show that total pressures at moderate depths are less than those computed on the fluid hypothesis, although at extreme depths the reverse may be true. Further, the Chicago clays are shown to act in connection with excavation work as if they have an angle of internal friction of 0°. It follows that any assumptions that neglect the significant soil properties are irrational and, although generally the accepted assumptions are on the safe side, occasions may arise wherein such assumptions would lead to unsafe designs. Mr. Peck's suggestions that the unconfined compression strength of the soil and the depth of the cut be used as factors in determining total lateral pressure, and that pressure diagrams be developed from this data, seem to offer a logical and relatively simple approach to the fixing of this very essential element of design.

Many of the other results of soil tests on the Chicago subway have proved of great value in establishing safe construction procedures. The rates of move-

ment of the clay in tunnel headings were observed by spearheads driven into the clay in advance of the headings, and from these measurements practical cycles of operations were established. Similarly, lateral movements were determined from so-called "squeeze" tests. By close observation of the extent of the movements and the time intervals involved, practical limits for spacing of lateral braces were fixed and measures taken for pre-stressing the tunnel linings to avoid undue or unsafe ground settlement.

In open-cut excavations on the Chicago subway, extended observations were made as outlined in detail in Mr. Peck's paper. This supplied the necessary data for the determination of the safe vertical spacing of the wales and bracing, the depth of penetration of the sheeting, and the magnitude of prestressing of the struts required to prevent untoward movements of the adjacent ground and to keep settlements within practical limits.

Although much of the experimental data obtained from soil tests on the Chicago subway were collected during the progress of the work, and were thus too late to affect the design, this branch of the engineering service was of great benefit on construction procedures. It is further felt that the results will be of inestimable value for future designs both to the city departments in new public works and also to the engineering profession as a whole. These tests not only furnish more exact knowledge of the physical properties of Chicago clays than has heretofore been available, but they also point the way to a more rational basis of design and to safer construction procedures.

L. G. LENHARDT,[22] M. AM. SOC. C. E.[22a]—The Central Avenue Tunnel, described by Professor Housel, was designed to convey a 60-in. steel water main beneath the classification yards and main-line tracks of the Michigan Central Railroad Company and two grade separations. There were twenty-two tracks at grade to be passed under and seven tracks carried on the separations, which supported the "hump" tracks of the Michigan Central classification yards. The volume of traffic on the tracks, the character of the soil, and probable future grade separation led to the decision of carrying this important transmission main beneath these obstacles in the tunnel.

The tunnel as constructed was some 1,200 ft long between center lines of shafts. The inside diameter was 9 ft 6 in. with an average wall thickness of about 20 in. Reinforcing steel was used which consisted of 1-in. square bars on the inside top and bottom quadrants of the lining and $\frac{3}{4}$-in. round bars on the outside at the spring-line quadrants. Spacing averaged about 12-in. centers. Longitudinal steel consisted of $\frac{3}{4}$-in. round bars at 18-in. centers.

The soil encountered at tunnel depths was a typical soft Detroit blue clay which is known as "squeezing" clay. Therefore the specifications required the work to be done with compressed air. With the pressures used there was generally little subsidence of the ground above the tunnel, and slight subsidence of grade-separation abutments was recorded. No particular construction difficulties were encountered other than the usual difficulties attendant

[22] Gen. Mgr. and Chf. Engr., Dept. of Water Supply, Detroit, Mich.

[22a] Received by the Secretary July 15, 1942.

on compressed air tunneling in soft blue clays where unremitting vigilance is the price of successful construction.

Mining operations were conducted during the day with a usual progress of 12 ft to 15 ft per shift. Crown planking was not used except beneath the railroad tracks where some eight to ten planks were carried. The concrete lining was placed, on the night shift, to within about 3 ft of the tunnel face so that at no time, except over week ends, would more than 3 ft of tunnel be exposed for a period of 24 hr. Over week ends the tunnel face was well breasted.

During the progress of the tunneling, grout was forced over the concrete bulkheads every other day through grout holes spaced at a maximum of 10-ft centers in the crown with additional grout piping at the upper quarter points where they were believed necessary. A few grout holes were also placed at the spring line and at the invert as checks; but no grout could be placed except through the crown grout holes and occasionally at the upper quarter points. When the tunnel lining operations were completed grout holes were cleaned out and the tunnel grouted to refusal.

It is the writer's opinion that the grouting tended toward uniformity of soil pressures during the early months. Grouting pressures did not find their ultimate release until some time after the tunnel was constructed. This was noticeable as holes drilled through the tunnel lining after construction showed water and air under pressure. This could only have originated from the grouting as there was no water encountered in the tunneling operations. Upon first making the tap outside of the lining the water was plainly under pressure but it was quickly relieved and these holes soon dried up.

The characteristic elliptical pressure observed in the later measurements was also observed in pressure cells installed in other tunnels during construction of the Springwells project.[23] Unfortunately access to these cells is no longer available due to the use of the tunnels.

The pressure cell measurements again emphasize the necessity for speed in soft ground tunneling operations. Further, the Central Avenue Tunnel is a relatively rigid structure and, as such, is essentially different from thin-walled structures, which, due to large deflections, utilize the passive resistance of the soil to maintain uniform pressures around their circumference.

Professor Housel has presented an ingenious analysis of an elastic structure in a plastic clay soil which cannot help but challenge attention. It should be stated that his analysis is based entirely on the concept of soft clay as a truly cohesive material. The fact that he closes his analyses through the various conditions of loading merits careful consideration.

GEORGE E. SHAFER,[24] ASSOC. M. AM. SOC. C. E.[24a]—From the standpoint of a practicing engineer as contrasted with the soil-research engineer, it is refreshing to read a paper on soil mechanics that is not filled with theoretical mathematical formulas and that can be followed intelligently by the engineer

[23] "The Springwells Filtration Plant," by Eugene A. Hardin, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), Fig. 1, p. 346.

[24] Chf. Engr., Armco Culvert Mfrs. Assn., Middletown, Ohio.

[24a] Received by the Secretary July 13, 1942.

who has the practical problem of determining the probable load that will be supported by an undergound structure such as a tunnel. Professor Housel has applied theory to a 10-yr experiment and has arrived at a workable solution to an age-old problem in which the structure to be built will be in plastic soil. This is a valuable contribution to the profession, as any one will appreciate who has had the problem of determining pressures on conduits installed by the tunnel method.

True, Anson Marston, Past-President and Hon. M. Am. Soc. C. E., and his associates at the Iowa Engineering Experiment Station have conducted research over a long period of years on pressures on conduits, and the result of that monumental work[25] has made the design of certain types of conduits a science instead of guesswork; but the Iowa research was concerned with conduits placed in open trenches or on top of the ground and then covered with earth, whereas Professor Housel's work deals with structures on which the load comes not from earth placed on or against the structure by man but by readjustment of the plastic earth itself when part of its support has been removed.

Engineers who encounter the problem of tunnel pressures for the first time are surprised to learn that tunnel structures are designed more by rule of thumb than by actual design data. The reason, of course, is that until quite recently very little was known as to the magnitude of the pressure to be supported, and still less as to the distribution of that pressure around the periphery of the tunnel.

There are two distinct parts to the problem of pressures on tunnels (as revealed in Professor Housel's paper): One is the initial pressure, which governs the design of the temporary lining of the tunnel; and the second is the maximum final pressure that will govern the design of the permanent lining.

Some years ago when the organization with which the writer is associated entered the metal tunnel liner field, the writer was amazed to discover that there was no rational basis on which to design liner-plates and no way but costly experience for the contractor to determine how much strength his plates would require. The practice was (and still is) to place the "burden of proof" on the contractor, who was expected to use his experience on other jobs to determine what liner-plates he should buy for this particular contract. He may buy heavier plates than necessary, in which case no one will ever know how much money might have been saved if science had replaced guesswork. If the contractor is optimistic, the chances are that the plates will be too light and he is then faced with all kinds of trouble such as the danger of physical injury to workmen, putting in compressed-air apparatus, getting structural reinforcement members, or trying to get the plate manufacturer to take back a few carloads of plates and supply heavier ones overnight.

There are, of course, certain limitations to the application of the method proposed by Professor Housel. In the first place, it applies only to soils in which the shearing resistance is a function of true cohesion and independent of internal friction. In the language of the practicing engineer, this means what

[25] *Bulletin No. 96*, Eng. Experiment Station, Iowa State College, Ames, Iowa.

is generally classified as clay, and except for tunnels in bedrock probably includes a large percentage of the cases encountered in practice. Then, as stated by Professor Housel, the pressure equations do not permit a direct solution without the introduction of simplifying conditions. If air is to be used, this is one of the simplifying conditions through the introduction of equal pressures at the top, side, and bottom of the tunnel. In making use of measured shear values, it is necessary to select what the author terms an "overload ratio." Fortunately, enough experience is already available to permit the engineer to establish this ratio with reasonable success.

The investigation has brought to light one phenomenon which may puzzle many, and that is the possibility of a decreasing load on the structure for the first three months of its life. However, that is not as important to the engineer or contractor responsible for the temporary lining as is the fact that the initial load on the temporary lining does not increase for this period, which should give sufficient time for placing the permanent tunnel structure.

The results on time required to reach final equilibrium check quite closely with the results of Dean Marston's experiments, previously mentioned,[25] although naturally the final loads could not be expected to be similar.

Given the shearing values of the soil (which, of course, will have to be determined from soil samples obtained at the site for that purpose) and sufficient other information to permit assigning a reasonably accurate "overload ratio," the engineer now has a new tool for use in designing both a temporary lining for a tunnel and the permanent structure as well.

Other applications of this same principle doubtless will be discovered in connection with other types of structures subjected to loads produced by earth masses in which the condition of equilibrium has been disturbed.

Frequently, one hears the criticism that the studies of soil mechanics research men are too theoretical and abstruse. Here is one excellent example of a practical application of soil mechanics, an application that can remove a lot of guesswork and save contractors and owners sizable sums of money.

M. E. CHAMBERLAIN,[26] ASSOC. M. AM. SOC. C. E.[26a]—During the construction of the Chicago subway, the writer as one of the consultants followed with interest and profit the soil investigations outlined in this Symposium. To state that construction methods during the building of these works were influenced, to a great degree, by the results of these investigations would be an understatement. The farsightedness of Messrs. Harrington and Burke in instituting these investigations contributed greatly to the safety of construction and economy. Careful examination of the results will make for greater safety and a considerable saving in future construction. Design of future construction, in this class of ground, may be approached with a much clearer understanding of the problems involved.

In the main, as noted, the material found north of the Chicago River is a stiff blue clay at the depths involved in the construction, and it is to the capacity of this material to arch and resist shear that the successful use of the so-called

[26] Maj., Corps of Engrs., U. S. Army, Sault Ste. Marie, Mich.

[26a] Received by the Secretary July 13, 1942.

"liner-plate" or "light-plate" method may be attributed. The upper stratum of hard yellow clay must have added little except weight to the overburden since, for years, it has been broken along the full length of the work during the laying of public utilities and other structures.

It is a matter of common knowledge that the stiff blue clays found in this region have a very strong structure, as evidenced by the many thousands of feet of small circular tunnels that have been built without temporary support. Some of these tunnels have had an outside diameter of as many as 12 ft. In these works the permanent lining is extended almost to the face of the excavation at all times, thus reducing to a minimum the amount of ground open at any one time. The time that any ground is without support is about the same as that required to place a temporary lining. The settlement on these projects is usually confined to an area of about twice the width of the structure, and the amount of settlement increased in the softer clays.

To combat this settlement a method of construction known as the liner-plate method was developed, in which light steel plates were placed in contact with, and jacked hard against, the ground as soon as it was exposed. The plates were supported first on the bench and then on a needle beam placed at or near the center of the tunnel. After the predetermined length of tunnel had been excavated and the temporary lining placed, the jacks were removed as the concreting advanced. Grouting of the permanent lining was completed daily to the end of the finished concrete lining. The outstanding feature of this method was that no part of the overburden was without support at any time prior to the completion of the permanent lining.

A large number of circular liner-plate tunnels have been constructed in clay and other materials in the twenty years 1922–1942 without any considerable amount of settlement, including the trunk sewer (16.5 ft in outside diameter) in the City of Detroit. In four of the Detroit tunnels there was only the slight uniform settlement of the pavement caused by the initial movement of the clay that had been found to be unavoidable in Chicago. It is interesting to note that, in Chicago also, the settlement was confined to an area of less than twice the width of the structure.

The use of the liner-plates became a very popular method of construction until the desire for long traveler forms for concrete, greater daily progress, and the application of plates to large-diameter tunnels caused the introduction of internal supporting ribs. The introduction of ribs, and the elimination of the complete 24-hr cycle of operation from the start of excavation to the completion of the permanent lining, changed the liner-plate method back to the old heading and bench method.

For a number of years the use of steel ribs with steel-plate lagging has been standard in Chicago. Many sewer tunnel sections of the Sanitary District have been built with complete success. Since the ground conditions in the northerly sections of the subway work varied but little from those which prevailed in a greater part of the Sanitary District, no sufficient reason could be advanced for not using a similar type of structure. The fact that the cross section of the subway tunnels was greater than the cross section of those pre-

viously built did present many problems, as the works were to be built in a thickly populated part of the city, and settlement had to be reduced to a minimum.

It was recognized that the use of any method of construction involving open heading and bench excavation depends in great measure on the capacity of the ground to arch and to support itself for at least a period of time necessary to the placement of the temporary lining. As no records of actual measurements of the magnitude or direction of the loads to be handled by the temporary lining were available, it was necessary to depend on the results observed in work previously constructed.

Work was started on contract S-5 (Fig. 1) in what was considered good clay. Steel ribs with liner-plates bolted to them, but set away from the clay to provide space for packing and grouting, were used. It became apparent in the early stages of this work that the ground was not being held during the several operations essential to the completion of the full rib. This fact contributed to excessive settlement and caused the development of weight that on several occasions resulted in severe distress of the ribs.

The aforementioned situation offered a fertile field for investigation, and actual measurements of the loads being developed were made. Careful study of these data made it possible to improve methods immediately, and on contract S-6, in clay of the same stiffness, it was possible to reduce settlement materially. The principal changes that were made on contract S-6 were the development of a plate that could be blocked or wedged to contact with the clay when placed, and the use of wall plates at the spring line. Of all the improvements, the most important was the use of the wall plates, as it was possible by means of jacks to place the full arch in bearing and hold it until the posts could be placed and wedged to the foot blocks. These changes became standard for the remaining work.

Many of the difficulties that now seem inherent in work of this character can be eliminated, or greatly reduced, by careful study of data such as those contained in this Symposium. It is to be hoped that similar investigations may be made in other classes of ground.

South of the Chicago River, after a careful examination of available data having to do with the subsurface conditions, and after evaluation of the possible difficulties due to heavy loading of the large structures along the line of the work, it was decided that the clay was too soft and plastic to permit the use of the method of construction that had been started on the northerly contracts. Therefore, the shield method was adopted, and it proved very satisfactory.

The clay in both the sections driven with shields was stiff enough to permit line and grade to be maintained with the jacks and a judicious use of taper rings. This fact made it possible to control the grade of the street by the amount and speed of the flow of the plastic clay through the pockets of the shield. The surface of the street was under continuous observation of level parties and, when the rise or fall of the street exceeded predetermined amounts, the heading was notified and corrections were made in the openings of the

shield. At a few points it was found that the clay was so soft that the surface would not respond to pressure, indicating that the use of any but the shield method would have allowed complete collapse.

It is interesting to note that, for almost the full length of the shield-driven tunnel, the volume of pea gravel used to fill the void caused by the tail of the shield exceeded the theoretical void. This would indicate that the action of the shield caused rupture of the clay even in these soft areas.

There is one conclusion drawn by Professor Terzaghi from the investigations on this work with which the writer is not entirely in accord. In the matter of future design of tunnel structures of the magnitude of the ones shown in Fig. 15, he shows a thin-walled structure, the design of which he states is based, in part, on the strength of the clay through which the tunnel is driven. If it were possible to state that the ground through which any considerable length of tunnel is to be driven will be uniform, and that in the future there will be no disturbance of the ground adjacent to the structure, the writer would be willing to explore the possibility of such a change with more enthusiasm. Consideration must also be given to the fact that concrete and other lining material may not measure up to laboratory standards. The personal equation which makes for perfection in workmanship must also be kept in mind at all times.

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DISCUSSIONS

# STRESS CONCENTRATIONS IN PLATES LOADED OVER SMALL AREAS

## Discussion

BY MESSRS. GEORGE R. RICH, DANA YOUNG, AND A. NÁDAI

GEORGE R. RICH,[21] M. AM. SOC. C. E.[21a]—The author has prepared a scholarly paper of great practical interest, in which functions of a complex variable are used to advantage. Since the analytical methods adopted will be valuable for general reference, it is appropriate to indicate that much of the qualifying material preceding Eqs. 107, 108, and 109 follows directly from the specification (from pure mathematics) of conditions that are necessary and sufficient for a function of a complex variable to be analytic.[22, 23] Included in this specification are the familiar Cauchy-Riemann equations which are necessary to verify that Eqs. 107, 108, and 109 satisfy basic Eqs. 104 and 105 when $p = 0$.

In connection with Eqs. 48 and 49, the writer was pleasantly surprised to find that the 1938 edition of the excellent tables of Jahnke and Emde is currently available.[24] However, it is believed that tables of the so-called "ker" and "kei" type of Bessel functions are more commonly used in the United States than those giving Hankel functions of the order of zero for complex values of the argument. Consequently, it may be useful as a reference to give the requisite transformation relation which is developed in the standard treatises:[25, 26, 27]

$$\tfrac{1}{2}\pi i H^{(1)}{}_0\left(\frac{r}{l}\sqrt{i}\right) = \ker\left(\frac{r}{l}\right) - i\,\mathrm{kei}\left(\frac{r}{l}\right) \ldots\ldots\ldots (148)$$

NOTE.—This paper by H. M. Westergaard, M. Am. Soc. C. E., was published in April, 1942, *Proceedings*.

[21] Chf. Design Engr., TVA, Knoxville, Tenn.

[21a] Received by the Secretary June 24, 1942.

[22] "Theory of Functions of a Complex Variable," by E. T. Copson, Oxford Univ. Press, 1935, p. 40.

[23] "The Theory of Functions," by E. C. Titchmarsh, Oxford Univ. Press, 1939, p. 64.

[24] "Tables of Functions with Formulæ and Curves," by Eugen Jahnke and Fritz Emde (B. G. Teubner, Leipzig and Berlin, and G. E. Stechert and Co., New York, N. Y.), 1938, p. 252.

[25] "Bessel Functions for Engineers," by N. W. McLachlan, Oxford Univ. Press, 1934, p. 170, and Table 10, p. 179.

[26] "A Treatise on Bessel Functions," by Andrew Gray, G. B. Mathews, and T. M. Macrobert, Macmillan and Co., Ltd., London, 1931, pp. 26, 58, and 59.

[27] "Theory of Functions of a Complex Variable," by E. T. Copson, Oxford Univ. Press, 1935, p. 313.

In "ker" and "kei" functions, Eqs. 48 and 49 become:

$$\zeta = \frac{P\,l^2}{4\,N}\,\mathrm{Re}\,H^{(1)}{}_0\left(\frac{r}{l}\sqrt{i}\right) = \frac{P\,l^2}{4\,N}\left(-\frac{2\,\mathrm{kei}\left(\frac{r}{l}\right)}{\pi}\right) \quad \ldots\ldots (149a)$$

and

$$\Delta\zeta = \frac{P}{4\,N}\,\mathrm{Im}\,H^{(1)}{}_0\left(\frac{r}{l}\sqrt{i}\right) = \frac{P}{4\,N}\left(-\frac{2\,\mathrm{ker}\left(\frac{r}{l}\right)}{\pi}\right) \quad \ldots\ldots (149b)$$

Modern tables of these functions,[25] with convenient asymptotic expansions[25, 26] for the larger values of the argument, are readily available. A thoroughgoing coordinate treatment of the general theory of Bessel functions is best obtained through the medium of complex variable theory and is preferably introduced by way of the Laplacian equation.[27]

The conversion of the Laplacian operator to cylindrical coordinates, Eq. 3, can be obtained, of course, by elementary methods; but, since the engineer assigned to analytical work has frequent occasion to effect a transformation of many similar types of differential invariants from Cartesian to orthogonal curvilinear coordinates, it may be stated in passing that such conversions are obtained with striking ease and directness by means of the methods of tensor analysis. For example, in the case of Eq. 3: The line element is—

$$\overline{ds}^2 = \overline{dR}^2 + R^2\,\overline{d\alpha}^2 \quad \ldots\ldots (150a)$$

and the non-zero components of the metric tensor are—

$$g_{11} = 1; \qquad g_{22} = R^2; \qquad g^{11} = 1; \qquad \text{and} \qquad g^{22} = \frac{1}{R^2} \quad \ldots (150b)$$

Using Eddington's[28] notation, the tensor expression for the Laplacian operator, valid in any system of coordinates, in space of any number of dimensions, is:[29]

$$g^{\mu\nu}\left(\frac{\partial^2\phi}{\partial x_\mu\,\partial x_\nu} - \{\mu\,\nu,\,\alpha\}\,\frac{\partial\phi}{\partial x_\alpha}\right)$$

The only non-zero value of the Christoffel three-index symbol in the foregoing case is $\{22, 1\} = -R$, and the conversion is immediate:

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\alpha^2} \quad \ldots\ldots (151)$$

The author states that the coefficient $K$ defined by Eqs. 11 and 21 is a logarithmic potential, and mentions electrical potential analogies that exist in connection with the derivation of Eq. 65. A familiar and perhaps closer analogy occurs in the theory of elasticity in which both direct and logarithmic potentials are used in calculating the displacements produced at any point in

[28] "The Mathematical Theory of Relativity," by A. S. Eddington, Cambridge Univ. Press, 1937, 2d Ed., pp. 55–64.

[29] "The Theory of Spherical and Ellipsoidal Harmonics," by E. W. Hobson, Cambridge Univ. Press, 1931, pp. 2–8.

a solid body by the application of pressure distributed over an area on the bounding plane.[30]

DANA YOUNG,[31] M. AM. SOC. C. E.[31a]—The determination of the stresses in a plate in the vicinity of a transverse concentrated load is one of the most troublesome problems in plate analysis. The ordinary theory of bending of plates is not valid for stresses at a point load, and it is necessary to correct the results in accordance with more exact theory or to take into consideration the fact that in most practical applications the load is actually distributed over a small area. A variety of special methods have been used in the past to obtain solutions for a limited number of cases. The author has succeeded in developing an original and powerful treatment of the problem which possesses great generality.

The sets of place coefficients and area coefficients given in the paper provide solutions for many useful cases. Data for simply supported and clamped rectangular plates of finite length were not given in the paper, and it may be of interest to derive the place coefficients for these cases.

For a rectangular plate with simply supported edges at $x = 0$, $x = l$, and $y = \pm \frac{1}{2} l_1$, and with a load $P$ uniformly distributed over a small circular area of radius $a$, whose center is at $x = v$, $y = 0$, the moments under the center of the load as given by S. Timoshenko[32] are:

$$m_x = \frac{P}{4\pi}\left[ (1+\mu) \log \frac{2\, l \sin \frac{\pi v}{l}}{\pi a} + 1 + \gamma_1 \right] \ldots\ldots\ldots (152a)$$

and

$$m_y = \frac{P}{4\pi}\left[ (1+\mu) \log \frac{2\, l \sin \frac{\pi v}{l}}{\pi a} + 1 + \gamma_2 - (1-\mu) \right] \ldots . (152b)$$

in which

$$\left.\begin{matrix}\gamma_1 \\ \gamma_2\end{matrix}\right\} = 2 \sum_{m=1}^{\infty} \frac{1}{m} \sin^2 \left( \frac{m \pi v}{l} \right) \left[ (1+\mu) (\tanh \alpha_m - 1) \mp \frac{(1-\mu)\, \alpha_m}{\cosh^2 \alpha_m} \right] \ldots (153)$$

and $\alpha_m = \dfrac{m \pi l_1}{2 l}$. Substituting Eq. 153 in Eqs. 152 and rearranging:

$$\left.\begin{matrix}m_x \\ m_y\end{matrix}\right\} = \frac{P}{4\pi} \left\{ (1+\mu) \left[ \log \left( \frac{2}{\pi} \sin \frac{\pi v}{l} \right) + \frac{1}{2} + \log \frac{l}{a} + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left( \sin^2 \frac{m \pi v}{l} \right) \right. \right.$$

$$\left. \left. (\tanh \alpha_m - 1) \right] \pm \frac{1-\mu}{2} \left[ 1 - 4 \sum_{m=1}^{\infty} \frac{1}{m} \left( \sin^2 \frac{m \pi v}{l} \right) \left( \frac{\alpha_m}{\cosh^2 \alpha_m} \right) \right] \right\} \ldots (154)$$

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[30] "The Mathematical Theory of Elasticity," by A. E. H. Love, Cambridge Univ. Press, 1934, 4th Ed., p. 192.

[31] Prof., Applied Mechanics, The Univ. of Texas, Austin, Tex.

[31a] Received by the Secretary June 29, 1942.

[32] "Über die Biegung der allseitig unterstützten rechteckigen Platte unter Wirkung einer Einzellast," by S. Timoshenko, *Der Bauingenieur*, Vol. 3, 1922, p. 51; or "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, pp. 156–167.

From Eq. 52, with $c = l$, it is noted that $K = \frac{1}{2} + \log \frac{l}{a}$ and $S = 0$. Then, comparing Eq. 154 with Eq. 13:

$$B = \log \left( \frac{2}{\pi} \sin \frac{\pi v}{l} \right) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left( \sin^2 \frac{m \pi v}{l} \right) (\tanh \alpha_m - 1) \ldots (155a)$$

and

$$C = 1 - 4 \sum_{m=1}^{\infty} \frac{1}{m} \left( \sin^2 \frac{m \pi v}{l} \right) \left( \frac{\alpha_m}{\cosh^2 \alpha_m} \right) \ldots\ldots\ldots (155b)$$

When $l_1$ is very large, $\tanh \alpha_m - 1 \leqq 0$, $\frac{\alpha_m}{\cosh^2 \alpha_m} \leqq 0$, and Eqs. 155 reduce to Eq. 39 $\left( \text{note that } v = \frac{l}{2} + u \text{ and } \sin \frac{\pi v}{l} = \cos \frac{\pi u}{l} \right)$. The series in the foregoing expressions for $B$ and $C$ converge rapidly. Numerical values for $B$ and $C$ for the case of a central concentrated load $\left( v = \frac{l}{2} \right)$ and several ratios $\frac{l_1}{l}$ are given in Table 3.

TABLE 3.—PLACE COEFFICIENTS FOR SIMPLY SUPPORTED AND CLAMPED RECTANGULAR PLATES WITH A CENTRAL CONCENTRATED LOAD

| $\frac{l_1}{l}$ | SIMPLY SUPPORTED | | CLAMPED | |
|---|---|---|---|---|
| | $B$ | $C$ | $B$ | $C$ |
| 1.0 | −0.6174 | 0 | −1.137 | 0 |
| 1.4 | −0.5002 | 0.5776 | −1.049 | 0.3968 |
| 2 | −0.4591 | 0.9065 | −1.044 | 0.5294 |
| ∞ | −0.4516 | 1.0000 | −1.044 | 0.538 |

Notes: $\frac{l_1}{l}$ = ratio of length of plate in $y$-direction to length in $x$-direction. Values for $B$ are based on $c = l$. Values for $\frac{l_1}{l_2} = \infty$ are taken from author's paper.

For a similarly loaded rectangular plate, with all four edges clamped, the moments at any point are found by superposing upon the moments for a simply supported plate the moments due to the distributed edge couples of the clamped plate. The solution for a clamped rectangular plate with a central concentrated load has been given,[33] and from this solution it may be shown that the corrections to be added to the place coefficients in Eqs. 155 to obtain the place coefficients for a clamped rectangular plate with a central concentrated load are:

$$B \text{ (correction)} = \frac{4 \pi}{P} \sum_{m=1,3,5 \ldots} \frac{(-1)^{\frac{m-1}{2}}}{2} \left( \frac{A_m}{\cosh \alpha_m} + \frac{B_m}{\cosh \beta_m} \right) \ldots (156a)$$

and

$$C \text{ (correction)} = \frac{4 \pi}{P} \sum_{m=1,3,5 \ldots} (-1)^{\frac{m-1}{2}} \left( A_m \frac{\alpha_m \tanh \alpha_m - 1}{\cosh \alpha_m} - B_m \frac{\beta_m \tanh \beta_m - 1}{\cosh \beta_m} \right) \ldots\ldots\ldots\ldots\ldots (156b)$$

in which $\alpha_m = \frac{m \pi l_1}{2 l}$, $\beta_m = \frac{m \pi l}{2 l_1}$, and $A_m$ and $B_m$ are numerical coefficients with values as given in the reference cited.[33]

[33] "Clamped Rectangular Plate with a Central Concentrated Load," by Dana Young, *Journal of Applied Mechanics*, Vol. 6, No. 3, September, 1939, p A-114; also Vol. 7, No. 1, March, 1940. p. A-43.

Numerical values for the place coefficients for the clamped plate are also given in Table 3.

A. NÁDAI,[34] ESQ.[34a]—The paper by Dean Westergaard on concentrated stresses in plates loaded under transversal forces which are distributed over comparatively small areas deserves to be studied most carefully by engineers. Those interested in the practical applications of the mathematical theory of bending of elastic plates must welcome this concise study of the effects of concentrated loads. It is an established fact that the foundations for the handling of the problems of the theory of flat plates have long been developed. For example, mathematicians have developed means for determining distributions of certain potential functions in space or over areas, the theory of which can furnish the basis for solving many plate problems. Despite these early developments (and also valuable recent advances made by engineers) for solving such problems, the practical applicability of the bending theory of elastic plates is still somewhat limited by certain facts. It is the intention of this discussion to point out at least a few of these facts, some of which were revealed in the paper.

The methods available for computing stresses in flat plates loaded by concentrated forces belong to this group of questions. It must be noted that certain difficulties that are encountered when the maximum bending stresses in such plates must be evaluated are not based on some inherent weaknesses of the plate theory. Rather, they are due to the fact that engineers dealing with these problems perhaps have not yet applied the special methods which are needed for solving such cases. The contrary is the truth concerning the aforementioned difficulties—namely, that the theory is able to furnish the required and entirely satisfactory means for finding the peak values of the stresses in plates loaded by concentrated forces.

It is well known that in the section in which a single transverse load is applied in a beam the bending moment has a discontinuity. In the bending theory of beams this discontinuity does not introduce any difficulties, and the bending stresses in the vicinity of the area in which a "concentrated load" is applied are computed from the same formula as the stresses in more distant points. The assumption (never mentioned) is that in the practical applications such a load necessarily should be distributed over a sufficiently large contact area, with the adjoining member of the construction transmitting the load, so that no further "local effects" around the pressure area need to be considered in most cases.

When the corresponding case of a concentrated load occurs in a plate, engineers can, and should, take somewhat the same attitude toward the plate theory. The extreme cases of a very strong load concentration in a small area can be excluded from consideration since they also introduce practical difficulties in construction. As Dean Westergaard states clearly, one should distinguish as to whether the concentrated load acts in the interior of the surface of the plate or near or along one of the edge sections. These two cases need somewhat

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[34a] Received by the Secretary July 15, 1942.

different methods of attack. The present discussion has been confined only to the first case.

Few instances are known to the writer in which authors who have dealt with similar cases have gone to the trouble of establishing the workable expressions needed for expressing the distribution of the bending stresses in the vicinity and within the area through which the concentrated load acts. Although the form of a solution of Lagrange's equation (Eq. 23) for a load which is concentrated in a mathematical point is known in the vicinity of this point, other important properties of the solution rarely have been visualized or discussed by those who make use of these developments in practical applications.

The static feeling of an engineer tells him that it should not, and does not, matter much how a load is actually distributed over an area, which is sufficiently small (not "too small" when compared in its dimensions with the thickness $h$ of the plate and not "too large" when compared with the smallest lateral dimension of the plate, measured parallel to its plane), if one wants to determine the stresses only in points of the plate, which are farther away from the loaded area. Practically very little or no use has been made of this good instinctive feeling in the mathematical applications of the plate theory. In other words, it is useful and helps in the discussion of the case if the load is replaced first by a single force concentrated in a mathematical point. Clearly this is a mathematical abstraction, but a very useful one, among the many other ones used, provided one confines it to its proper limits.

Engineers probably have not been conscious of the fact that finding the proper integral of Lagrange's equation of the plate theory under the given boundary conditions which corresponds to the case of the single force concentrated in a point is the first necessary step toward establishing the expressions required for a computation of the peak stresses also near the finite loaded area. Experience has shown that the deflections and the form of the elastic surface can be computed at any point, including the area loaded, by using this solution.

In the mathematical theory of the distribution of potentials around given bodies or areas, use is made of similar ideas when potential functions are introduced which (1) satisfy the differential equation of the potential, (2) become finite in a single point in a required manner, and (3) satisfy the prescribed boundary conditions along the surface of the body or along curves in the case of two-dimensional problems. These functions are there known as "Green's functions," which solve special cases of boundary conditions. The bending theory of flat plates is closely related to the theory of logarithmic potentials. Evidently, finding the solution for a given contour line of a plate under given boundary conditions and carrying a single force in an arbitrary point is identical with this problem of finding the Green function of Lagrange's plate equation of the fourth order, satisfying given boundary conditions. The computation of the stresses near or within the area in which a concentrated force acts in a plate thus must consist first of a determination of the Green function corresponding to the load concentrated in the center of the loaded area.

After the solution for a single concentrated force in the center of the loaded area has been found or constructed by the methods which are available for such cases the three "place coefficients," $B$, $C$, and $D$, introduced by the author,

can be computed readily by comparing it with Eq. 1. It is known, however, that "the logarithmic tube" which expresses the bending stresses in a plate near the point of application of the load increases to infinity near this point, and expressions based on the load concentrated in a point fail at and near the point of application. The author is to be commended for showing clearly that in the most general case three additional functions, $K$, $S$, and $T$, must be constructed as defined in Eqs. 11 and 12. These functions are needed if the bending stresses are to be computed near or within the loaded area. They need to be computed only once for a given pressure distribution and shape of the loaded area and are not dependent on the shape of the contour line of the plate. After these three functions have been constructed, the peak stresses, which are usually found within the loaded area of the plate, can be computed. It is interesting to note that in a few cases in the plate literature evidently these integrals have been constructed implicitly when the complete solutions were found by the ordinary methods of integration.

It is noted that the author has made frequent use of expressions containing functions of the complex variable $z = x + i\,y$. The theories of plane stress and of the bending of plates are both based on the biharmonic equation and closely related to the theory of functions. In plane hydrodynamics when certain patterns of potential flow are discussed the same methods are used.

In this connection perhaps the attempts of M. Bergstraesser,[35] in his doctor's thesis of 1928, may deserve mention. He used the methods of conformal mapping for the numerical construction of the bending moments in parallel strips and in rectangular plates loaded by a single force and satisfying the Navier boundary conditions. His efforts were directed toward the numerical determination of Green's function for the moment sum. The results of his elaborate computations, which were presented in curves plotted for the bending moments $m_x$ and $m_y$ and for the twisting moment $m_{xy}$, should become even more valuable today if the three place coefficients, $B$, $C$, and $D$, introduced by Dean Westergaard, were computed for the various cases which were discussed in this thesis.

Correction for *Transactions:* In April, 1942, *Proceedings*, page 516, Eqs. 38 and 39, change "$\frac{\pi u}{2l}$" to "$\frac{\pi u}{l}$" in three places.

[35] "Versuche mit freiaufliegenden rechteckigen Platten unter Einzelkraftbelastung," by M. Bergstraesser, V.D.I. *Forschungsarbeiten*, Heft 302, 1928.

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DISCUSSIONS

# EARLY CONTRIBUTIONS TO MISSISSIPPI RIVER HYDROLOGY

## Discussion

BY MESSRS. ROBERT FOLLANSBEE, AND R. W. DAVENPORT

ROBERT FOLLANSBEE,[19] M. AM. SOC. C. E.[19a]—Although the early contributions to Mississippi River hydrology were made chiefly by the Army engineers, the Navy made its contribution even before the classic work of Humphreys and Abbot. Acting under instructions from the Secretary of the Navy, Lt. Robert A. Marr, master of the Memphis Navy Yard, investigated the discharge of the Mississippi River at Memphis from April to July, 1849, and from March, 1850, to February, 1851. He made measurements by means of floats, and deduced the mean velocity from surface observations by means of floats sunk to various depths. The reduction factor was slightly more than 10%. He also obtained daily gage heights, and from these gage heights and the results of the discharge measurements he computed the daily discharge for the period covered. A description of this work was presented by the Washington Astronomical and Meteorological Observatory.[20]

The author refers to the earnest approaches made by the early contributors to Mississippi River hydrology toward the more rigorous standards developed during recent years. In that connection it may be noted that Humphreys and Abbot's study of the vertical-velocity curves obtained during their investigations indicated that the velocity at different depths varied as the abscissas of the parabola which had its $x$-axis parallel to and below the water surface. However, they did not pursue further the study of the properties of the parabolic curve. That was reserved for a British engineer, Maj. Allen Cunningham, who, during the 1870's, made 565 sets of vertical-velocity curves on the Ganges Canal in India. He concluded that, by considering the vertical-velocity curve as a common parabola, its properties indicated that the mean velocity equals $\frac{1}{2}$ (velocity at 0.211-depth + velocity at 0.789-depth).[21] These are practically

NOTE.—This paper by C. S. Jarvis, M. Am. Soc. C. E., was published in March, 1942, *Proceedings*.

[19] Dist. Engr., U. S. Geological Survey, Denver, Colo.

[19a] Received by the Secretary June 26, 1942.

[20] "Observations on the Mississippi River at Memphis, March 1850 to March 1851," by Robert A. Marr, Washington Astronomical and Meteorological Observatory, Vol. 3.

[21] "Recent Hydraulic Experiments," by Allen Cunningham, *Proceedings*, Inst. C. E., Vol. 71, London, 1883.

the 0.2-depth and the 0.8-depth. It was not until 1906, however, when H. K. Barrows, M. Am. Soc. C. E., then a member of the U. S. Geological Survey, made a study of vertical-velocity curves and found that their form was that of a parabola, the mean ordinate of which was closely the average of the ordinates at the 0.2-depth and the 0.8-depth, that the procedure of measuring the velocity of these depths and taking the average as the mean velocity was adopted by the U. S. Geological Survey as its standard.

R. W. DAVENPORT,[22] M. AM. SOC. C. E.[22a]—The author's research in the early records of the Mississippi River constitutes a valuable service in furnishing a "panoramic view" of the flow of this dominant waterway for a period of considerably more than a century.

The long-time compilations of runoff presented in the paper are subject to use in a variety of ways, not only for fixing the mean runoff over a long period of years but for studies of trends, and variation in magnitudes, for possible correlation with other factors. The author has included comparisons of deviations of runoff with those of rainfall. He also has in mind that the data may form a foundation on which to assemble, compare, and interpret land-use information.

It is particularly desirable in such uses of runoff data that the relative refinement of the data be known so that one may consider the extent, if any, to which evidence of relationships, apparent or sought for, may be influenced by fortuitous or systematic errors. It is not uncommon for such errors to obscure or to invalidate the significance of the evidence. It is desirable, as the author indicates, to suggest modifications that will make the results more nearly what is required for such uses. It is also very essential to know the inherent limitations of the results so that they may not be applied unwisely. If the data are satisfactorily free of such errors, the further question should be considered as to whether the significance of the factor investigated, such as land use, may be masked by other factors, particularly those of climatic variation.

The writer has been greatly impressed with the tremendous quantity of data concerning the history and behavior of the Mississippi River and feels quite humble in discussing a subject of which his knowledge encompasses comparatively little. The points raised herein are not intended to be in the way of criticisms, but more in the way of questions on which it is hoped the author may throw light.

The longest record developed is that of the Mississippi River at Natchez from 1817 to 1938, a period of 122 years, the results being shown in Tables 5 and 6. The technique used in the determination of the record in the earlier years may be learned by a perusal of *Professional Paper No. 13*,[23] which presents results for 33 out of the 42 years from 1818 to 1860. These records apparently were based on stages prior to 1848 solely at Natchez, and subsequently mostly at Carrollton and Donaldsonville, La. There seems to be very little information for appraising the accuracy of the basic stage records.

[22] Hydr. Engr. in Chg., Div. of Water Utilization, U. S. Geological Survey, Washington, D. C.

[22a] Received by the Secretary July 29, 1942.

[23] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876.

The stage-discharge relation for the Natchez station was determined from discharge measurements in January and February, 1858, at Natchez, and for the remainder of the year it was determined at Vicksburg and transferred to Natchez. In developing the relation a mean curve was apparently drawn through the loops described by the individual measurements of rising and falling stages. The inaccuracy involved in the determination of the discharge for a given day would tend to be compensative in computing the discharge for the year. The mean annual discharge was determined essentially by application of the twelve mean monthly stages of the year to the rating table, adding the results, and dividing by twelve.

The discharge determinations consisted of velocity measurements made by means of timing double floats, the most common form being a surface float of cork to which was attached a paint keg, the depth of submergence of which was fixed by a cord attached to the cork. The kegs were passed at all depths from the bottom to the top, but a uniform depth of 5 ft seems to have been adopted for general use. The mean velocities of the floats were determined for divisions 200 ft in width, and the products of the mean surface velocity and the areas of the cross sections for such divisions were computed and added for the entire cross section to give the "approximate discharge." Soundings were made in the vicinity of Natchez in January and in the vicinity of Vicksburg in February, September, and December during 1858, which apparently gave the basis for the areas of cross sections to be used in the discharge computations.

The "approximate discharge" was corrected by a ratio designed to adjust the float velocity to the mean velocity in the section and also to take account of the prevalent condition of the wind. Similarly, the discharges at Carrollton and Donaldsonville were computed from measurements made in 1851 and 1852.

This early river measurement technique, briefly summarized herein but described fully in *Professional Paper No. 13*,[23] was unquestionably the best afforded by the knowledge and facilities of the time and was conducted with supreme care. It may be compared with that used at the present time by the Geological Survey for the gaging station at Vicksburg:[24]

> "Discharge Measurements.—Discharge measurements are made from the downstream side of the highway section of the bridge. The highway section is on the upstream side of the bridge, and the railway section on the downstream side. The metering equipment is lowered through an opening between the floors of the highway and railway. The equipment is operated by means of a reel and boom mounted on a low-wheeled truck frame. The truck frame also carries a four-cylinder automobile engine which supplies power for operating the metering equipment and moving the truck from point to point along the bridge. Depth of water and position of meter are shown by a depth indicator attached to the reel. During medium and high stages the depth of water and velocity are such that the metering equipment is carried downstream and the indicated depth is too large. Conversion to true depth is effected by use of two correction tables, one of which gives the correction to slant length of supporting line above water surface, and the other gives the correction to curved length of line below water surface. The use of each table requires a measurement of the vertical angle in the supporting line above the water surface, and this

[24] *Water-Supply Paper No. 747*, U. S. Geological Survey, Washington, D. C., 1933, p. 16.

measurement is accomplished by use of a protractor applied to the line just below the bridge floor. The corrections to indicated depth are lessened by the use of large sounding weights. The maximum sounding weight used at the Vicksburg station is 300 pounds.

. . . . . .

"Determination of Discharge.—Daily discharge determined by shifting-control method. Changes in stage-discharge relation determined by frequent measurements."

From the establishment of the Vicksburg gaging station in 1930 to the end of the water year 1941, 1,524 discharge measurements have been made at this station, generally two to three days apart, to record the vagaries of the flow of the river. Fig. 5 illustrates the plotting of discharge measurements in 1938 in

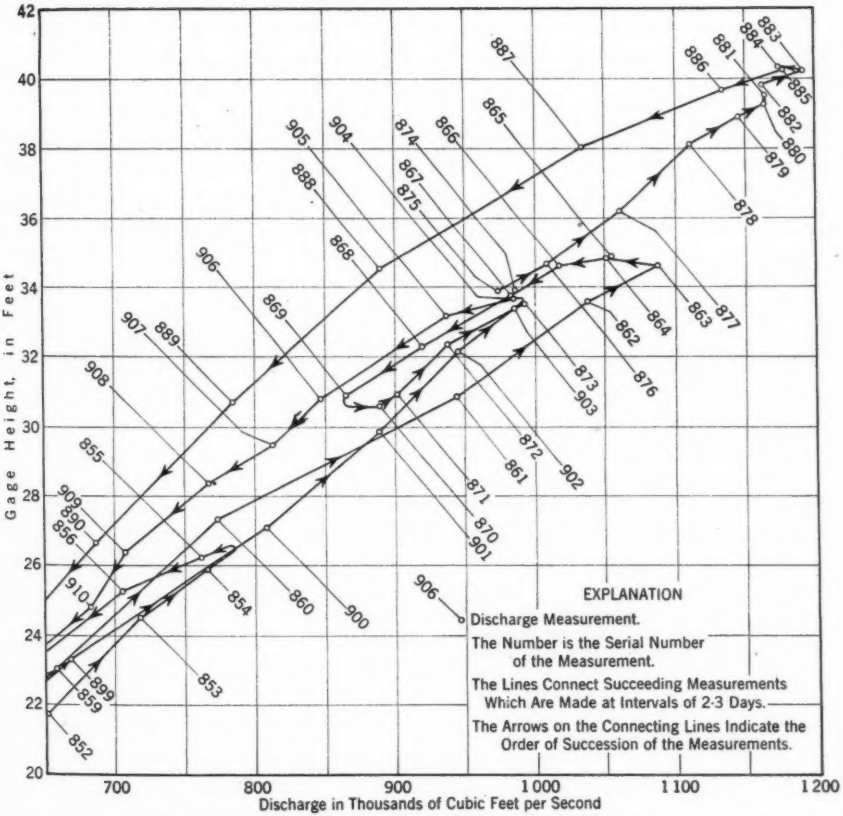
Gage Height, in Feet
Discharge in Thousands of Cubic Feet per Second
EXPLANATION
Discharge Measurement.
The Number is the Serial Number of the Measurement.
The Lines Connect Succeeding Measurements Which Are Made at Intervals of 2-3 Days.
The Arrows on the Connecting Lines Indicate the Order of Succession of the Measurements.

FIG. 5.—DISCHARGE MEASUREMENTS, MISSISSIPPI RIVER AT VICKSBURG, MISS., 1938, U. S. GEOLOGICAL SURVEY

excess of 700,000 cu ft per sec. The writer is unable to state how typical this period may have been. However, the plotted points suggest the complications involved in the determination of the stage-discharge relation for application to

the stage records. Soundings are made for each discharge measurement and the cross sections thus determined indicate considerable variation during short periods. Comparisons of areas of cross sections for gage heights of 28 to 30 ft from soundings from 1933 to 1939 indicate an increase in area of about 10%. This increase may have been associated with the effects of cutoffs below Vicksburg. However, it illustrates another changing aspect of the behavior of the Mississippi River.

The application of the stage-discharge relation as determined for Natchez in 1858 involves questions of reliability of stage records, the accuracy of the compensative adjustment by use of a mean curve drawn through rising and falling branches of the plotted measurements, uncertainty as to cross-sectional areas applicable at the time of a measurement, and other matters pertinent to a pioneer technique. Much more serious than these questions, however, are those pertinent to the application of the 1858 stage-discharge relation preceding the period in which the determining observations were made as far back as 1818. Illustrations in "The Improvement of the Lower Mississippi River for Flood Control and Navigation"[25] delineate pronounced progress in channel changes in the Mississippi River below Vicksburg and Natchez between 1820 and 1861 and also, apparently, in the construction of levees.

Fig. 2, which shows rating curves of the Mississippi River at Vicksburg from 1858 to 1937, indicates that in that period the mean rating curves that have been defined have varied within limits approximating $\pm 5\%$. It is apparently not demonstrated, however, that changes influential on the Vicksburg-Natchez section of the river may not have been substantially more operative in the period 1817 to 1858 than in the later period.

The critical feature in respect to the use of the earlier records is the cloud of uncertainty resting over their reliability. The author recognizes this situation in his supposition that modifications may be suggested that will make the results more nearly what is required for the analysis of land-use information. It is notable that the Mississippi River Commission has not included such modifications in its compilations of past records—for what reason the writer is not informed. In their present status, it is important to caution earnestly that in any applications of the records their inherent nature as respects accuracy be kept in mind.

[25] "The Improvement of the Lower Mississippi River for Flood Control and Navigation," prepared by D. O. Elliott, under the direction of T. H. Jackson, U. S. Waterways Experiment Station, Vicksburg, Miss., Vol. III, May 1, 1932, Plates VI(f), VI(g), VI(h), and XXXII.

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DISCUSSIONS

# BASIC ECONOMIC CONSIDERATIONS AFFECTING SINGLE AND MULTIPLE PURPOSE PROJECTS

## Discussion

BY G. B. PILLSBURY, M. AM. SOC. C. E.

G. B. PILLSBURY,[2] M. AM. SOC. C. E.[2a]—A procedure is presented in this paper for evaluating the economic benefits from proposed projects intended wholly or partly for flood prevention. The applicability of the procedure obviously depends upon the purpose that the investigation is to serve. An evaluation of flood-prevention benefits conceivably might be made for a large landholder, or group of landowners, to determine whether the construction of works for the protection of the property would be profitable as a private enterprise; or it might be undertaken for a municipality or flood control district to determine whether the local benefits would justify the local taxpayers in undertaking flood protection works; or it may be undertaken to determine the economic justification from the viewpoint of the adoption of a federal flood control or multiple purpose project. The procedure presented in the paper evidently has in view this third purpose.

A prudent, or even an imprudent, private owner would be moved principally by the anticipated enhancement of the value of his property in deciding upon the advisability of investing his own funds in flood control works. This enhancement might be credited with the flood losses prevented, which are listed as Item 2(*a*) in the paper, as well as with the more profitable utilization of the land after protection, listed as Item 2(*b*), but the latter is usually the compelling consideration. In any case, he would scarcely undertake any considerable expenditure unless at least the paper profits showed a wide margin over the estimated cost. The close figuring shown in the author's example ordinarily would not have much financial appeal.

Flood control works, whether in the nature of storm drains or more extensive enterprises, ordinarily are undertaken by municipalities or local flood control districts on broad grounds of public necessity or convenience, rather than because of a detailed evaluation of financial benefits. Generally, such works are adopted either to rectify an inconvenient or intolerable situation

NOTE.—This paper by Otis M. Page, M. Am. Soc. C. E., was published in March, 1942, *Proceedings*.

[2] Brig.-Gen., Corps of Engrs., U. S. Army (*Retired*), Ross, Calif.; formerly Asst. Chf. of Engrs.

[2a] Received by the Secretary June 10, 1942.

that has grown up because of municipal development, or to prevent the recurrence of a recently experienced flood disaster. Such works are built to satisfy an insistent public demand, and a close figuring of financial benefits is not often required. In exceptional cases, to be sure, legal requirements may entail the assessment of benefits against the properties affected, and the considerations presented by the author might be applicable to the preparation of the assessment roll.

In the investigation of a proposed federal flood control project, an evaluation of the benefits is required by the Flood Control Act of June 22, 1936, in which Congress laid down the policy:

> "* * * that the Federal Government should improve or participate in the improvement of navigable waterways or their tributaries, including watersheds thereof, for flood control purposes, if the benefits to whomsoever they may accrue are in excess of the estimated costs and if the lives and social security of people are otherwise adversely affected."

Sound policy undoubtedly requires that public works be undertaken only when the benefits exceed the estimated cost; but these benefits are not necessarily, nor even generally, material ones which can be evaluated by an engineering computation. The more important works contemplated by the Federal Flood Control Act are those for the prevention or mitigation of a recurrence of destructive floods which have caused loss of life as well as of property, have produced widespread inconvenience and discomfort because of the interruption of rail and highway traffic and other public services, and have inflicted suffering and misery upon the considerable number of persons who were driven from their homes. The writer does not believe it to be the intent of Congress that the justification for such projects should rest on a nice evaluation of the material benefits, but rather that the rule of reason be applied to determine, on general grounds, whether the value of benefits which cannot be definitely computed in money will exceed the estimated cost.

On the other hand, the original Flood Control Act included the authorization of a number of projects intended to remove a continuing flood hazard to agricultural lands on which the local inhabitants had exhausted their resources in attempted protection, or were unable to raise funds for the necessary works. The procedure described by the author would appear particularly applicable to the investigation of proposed projects of this character.

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DISCUSSIONS

# MOMENT BALANCE: A SELF-CHECKING ANALYSIS OF RIGIDLY JOINTED FRAMES

## Discussion

BY MESSRS. FREDERICK S. MERRITT, AND RALPH W. STEWART

FREDERICK S. MERRITT,[11] JUN. AM. SOC. C. E.[11a]—The method of analyzing continuous structures by moment balance has two main features to recommend it. First, by enabling designers to estimate final moments, considerable work may be saved. Second, its self-checking character may also save time, in that errors may be either immediately located or neutralized by further balancing. Accounting for sidesway moments is as much of a handicap in other methods as it is in this.

The method of moment balance may be applied in a somewhat different form with the advantage of ease of generalization and saving in computation. Let: $\alpha$ = six times the rotation at one end of a simply supported beam of unit length and least $K = \frac{I}{L} = 1$ due to a unit moment applied at that end; $\beta$ = six times the rotation at one end of this beam due to a unit moment applied at the other end; $K$ = the ratio of least moment of inertia to span length; $M$ = actual end moment minus fixed-end moment; and $\phi$ = the rotation at the end of any beam. Clockwise moments will be considered positive and counterclockwise moments negative. Then, referring to Fig. 1, in general, neglecting deflection at beam ends:

$$6\,E\,K_{ON}\,\phi_O = \alpha_{ON}\,M_{ON} - \beta_{ON}\,M_{NO} \quad \ldots\ldots (14a)$$

$$6\,E\,K_{OP}\,\phi_O = \alpha_{OP}\,M_{OP} - \beta_{OP}\,M_{PO} \quad \ldots\ldots (14b)$$

$$6\,E\,K_{OQ}\,\phi_O = \alpha_{OQ}\,M_{OQ} - \beta_{OQ}\,M_{QO} \quad \ldots\ldots (14c)$$

and

$$6\,E\,K_{OR}\,\phi_O = \alpha_{OR}\,M_{OR} - \beta_{OR}\,M_{RO} \quad \ldots\ldots (14d)$$

NOTE.—This paper by R. J. Cornish, Esq., was published in May, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1942, by Messrs. D. D. Matthews, and William A. Larsen; and June, 1942, by Messrs. A. A. Eremin, L. E. Grinter, and B. J. Aleck.

[11] Structural Designer, Gibbs & Hill, New York, N. Y.

[11a] Received by the Secretary June 10, 1942.

TABLE 3.—COMPUTATION SHEET

| No. | Quantity | Joint A | | | Joint B | | | | Joint C | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Member AD | Member AB | Σ | Member BA | Member BE | Member BC | Σ | Member CB | Member CF | Σ |
| .... | $K$ | 3 | 5 | 8 | 5 | 4 | 5 | 14 | 5 | 3 | 8 |
| .... | $M^F$ | .... | −8.4 | −8.4 | +2.4 | .... | .... | +2.4 | .... | .... | .... |
| | | | | | (*a*) First Balance | | | | | | |
| 1 | $\frac{1}{2} M'$ | +0.5 | +0.1 | +0.6 | +2.5 | +1.5 | 0 | +4.0 | −1.2 | −1.0 | −2.2 |
| 2 | $m$ | +2.9 | +4.9 | −7.8 | −2.3 | −1.8 | −2.3 | +6.4 | +1.4 | +0.8 | −2.2 |
| 3 | Change | +3.4 | +5.0 | .... | +0.2 | −0.3 | −2.3 | .... | +0.2 | −0.2 | .... |
| Final moment | | +3.4 | −3.4 | .... | +2.6 | −0.3 | −2.3 | .... | +0.2 | −0.2 | .... |

Divide each equation by its value of $\alpha_O$ and let $S = \frac{K}{\alpha}$ and $C = \frac{\beta}{\alpha}$. Note that $C$ is the carry-over factor toward O for the beam, considering it as fixed-ended. Summing Eqs. 14 yields:

$$6\,E\,\phi_O\,\Sigma S_O = \Sigma M_O - \Sigma(C\,M') \quad \ldots\ldots (15)$$

in which $M'$ is the change in the fixed-end moments at the far ends. Since $-\Sigma M_O = \Sigma M^F{}_O$ (the sum of the fixed-end moments at O due to loads)—

$$6\,E\,\phi_O\,\Sigma S_O = -[\Sigma M^F{}_O + \Sigma(C\,M')] \quad \ldots\ldots (16)$$

Substituting Eq. 16 in Eq. 14*a*:

$$M_{ON} = C\,M_{NO} - \frac{S_{ON}}{\Sigma S_O}[\Sigma M^F{}_O + \Sigma(C\,M')] \quad \ldots\ldots (17)$$

Using Eq. 17, the method of moment balance may be applied to continuous frames containing beams of variable moment of inertia. The final moments are obtained by adding the results of the last balance to the fixed-end moments. For beams of constant moment of inertia, Eq. 17 becomes:

$$M_{ON} = \tfrac{1}{2} M_{NO} - \frac{K_{ON}}{\Sigma K_O}(\Sigma M^F{}_O + \tfrac{1}{2}\Sigma M') = \tfrac{1}{2} M_{NO} - m \quad \ldots\ldots (18)$$

in which

$$m = \frac{K_{ON}}{\Sigma K_O}(\Sigma M^F{}_O + \tfrac{1}{2}\Sigma M') \quad \ldots\ldots (19)$$

Using Eq. 18, the frame in Fig. 2 was analyzed for a comparison of results (see Table 3). The same changes in fixed-end moments as assumed by the author were also taken as a preliminary estimate. Joint A was balanced first. One half the change at the far ends was recorded for AD and AB and the sum placed under the sigma. This sum was then added to the fixed-end moment (in this case zero), and the result, entered in Table 3, was distributed, with sign changed, to AD and AB in proportion to their $K$-values. The change in the fixed-end moment was obtained by adding the two values entered for

TABLE 3.—(*Continued*)

| No. | Quantity | Joint D | | | | Joint E | | | | | Joint F | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Member DA | Member DG | Member DE | Σ | Member ED | Member EB | Member EH | Member EF | Σ | Member FE | Member FC | Member FK | Σ |
| .... | $K$ | 3 | 4 | 6 | 13 | 6 | 4 | 5 | 6 | 21 | 6 | 3 | 4 | 13 |
| .... | $M^F$ | .... | .... | .... | .... | .... | .... | .... | −11.5 | −11.5 | +5.2 | .... | .... | +5.2 |
| | | (*a*) First Balance | | | | | | | | | | | | |
| 1 | $\frac{1}{2}M'$ | +1.7 | −0.5 | +2.0 | +3.2 | +0.3 | −0.2 | +0.8 | −0.6 | + 0.3 | +1.3 | −0.1 | −0.5 | +0.7 |
| 2 | $m$ | −0.7 | −1.0 | −1.5 | +3.2 | +3.2 | +2.1 | +2.7 | +3.2 | −11.2 | −2.7 | −1.4 | −1.8 | +5.9 |
| 3 | Change | +1.0 | −1.5 | +0.5 | .... | +3.5 | +1.9 | +3.5 | +2.6 | .... | −1.4 | −1.5 | −2.3 | .... |
| Final | moment | +1.0 | −1.5 | +0.5 | .... | +3.5 | +1.9 | +3.5 | −8.9 | .... | +3.8 | −1.5 | −2.3 | .... |

each member. If the final moment were desired at this stage, the addition would include the fixed-end moment, and the change in moment would not be entered. It will be noticed that at the end of the first balance results agreeing with those of the author were obtained.

The method of moment balance has the same disadvantage in practical application as other methods of analysis by converging increments—that is, in locating loads for maximum moment. Because of the unknown end conditions in interior spans, it is difficult to apply these methods to problems of moving loads.

Ralph W. Stewart,[12] M. Am. Soc. C. E.[12a]—The "Synopsis" of this paper states that the trial and revision procedure which it describes is rapid and that the arithmetical work is almost entirely self-checking. These claims are attractive and led the writer to experiment some with the author's demonstration problem shown in Fig. 2.

Table 4 shows the results of three independent solutions of this problem, as well as the final moments computed by the author. For the practical design of a building frame of this type it is necessary to consider various combinations of loading, including wind or earthquake loads. This requires that the frame be solved for the effect of a fixed-end moment at each of its joints. The first solution in Table 4 gives the effect of each individual fixed-end moment and is carried to an accuracy of 0.01 ft-ton. This first solution was performed without the use of simultaneous equations and without the use of any kind of successive approximations.

The solution by H. O. Springer (see Fig. 9) was performed by conventional end-moment distribution in 50 min, including the time required to make the sketch and write on it the stiffness factors and fixed-end moments. The solution by L. Blume (see Fig. 10), which is of the same order of accuracy as the author's two-place solution, required only 15 min after the sketch showing stiffnesses and fixed-end moments was prepared.

[12] Engr. of Bridge and Structural Design, City of Los Angeles, Los Angeles, Calif.

[12a] Received by the Secretary July 6, 1942.

TABLE 4.—COMPARISON OF FOUR INDEPENDENT SOLUTIONS (see Fig. 2)

| Description | Joint A | | Joint B | | | Joint C | |
|---|---|---|---|---|---|---|---|
| | AD | AB | BA | BE | BC | CB | CF |
| Effect of fixed-end moment, AB | −3.23 | +3.23 | −1.74 | +0.81 | +0.94 | +0.19 | −0.19 |
| Effect of fixed-end moment, BA | −0.20 | +0.20 | −1.83 | +0.87 | +0.96 | +0.20 | −0.20 |
| Effect of fixed-end moment, EF | +0.01 | −0.01 | +0.38 | −0.77 | +0.38 | −0.01 | +0.01 |
| Effect of fixed-end moment, FE | −0.03 | +0.03 | +0.11 | −0.02 | −0.09 | −0.33 | +0.33 |
| Combined moment | −3.45 | +3.45 | −3.08 | +0.84 | +2.19 | +0.05 | −0.05 |
| R. J. Cornish | 3.5 | 3.5 | 3.0 | 0.8 | 2.2 | 0.1 | 0.1 |
| H. O. Springer | 3.40 | 3.40 | 3.09 | 0.88 | 2.21 | 0.01 | 0.01 |
| L. Blume | 3.4 | 3.4 | 3.0 | 0.8 | 2.2 | 0.1 | 0.1 |

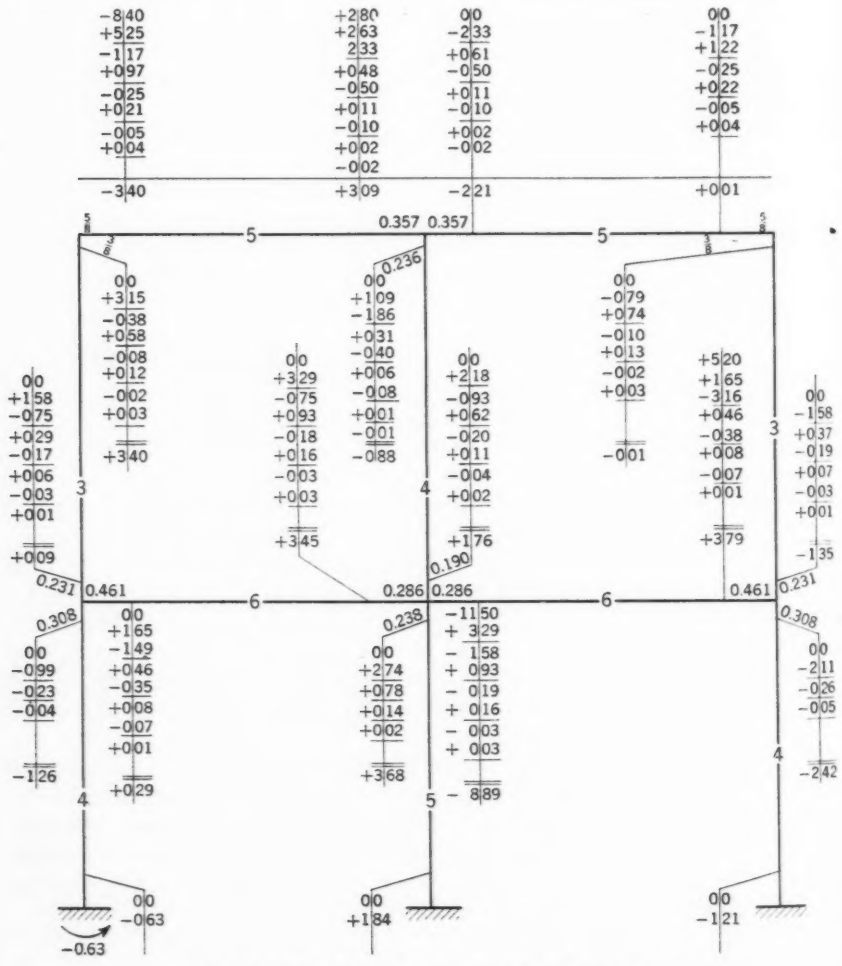

FIG. 9.—FIFTY-MINUTE SOLUTION BY H. O. SPRINGER

TABLE 4.—(*Continued*)

| Joint D | | | Joint E | | | | Joint F | | |
|---|---|---|---|---|---|---|---|---|---|
| DA | DG | DE | ED | EB | EH | EF | FE | FC | FK |
| −1.29 | +0.57 | +0.72 | +0.16 | +0.27 | −0.22 | −0.21 | −0.02 | −0.05 | +0.07 |
| −0.07 | +0.06 | +0.01 | −0.11 | +0.36 | −0.13 | −0.11 | +0.01 | −0.07 | +0.06 |
| +0.34 | +0.58 | −0.92 | −3.16 | −2.18 | −2.99 | +8.33 | −0.92 | +0.34 | +0.58 |
| +0.03 | +0.07 | −0.10 | −0.35 | −0.21 | −0.33 | +0.89 | −2.85 | +1.15 | +1.70 |
| −0.99 | +1.28 | −0.29 | −3.46 | −1.76 | −3.67 | +8.90 | −3.78 | +1.37 | +2.41 |
| 1.0 | 1.3 | 0.3 | 3.4 | 1.8 | 3.6 | 8.8 | 3.8 | 1.4 | 2.4 |
| 0.99 | 1.26 | 0.29 | 3.45 | 1.76 | 3.68 | 8.89 | 3.79 | 1.35 | 2.42 |
| 1.0 | 1.3 | 0.3 | 3.4 | 1.7 | 3.6 | 8.9 | 3.7 | 1.4 | 2.3 |

The author deserves credit for having made a scholarly investigation of the analysis of rigidly jointed frames. However, between the superior speed of the conventional end-moment distribution solution and the superior quality of the first solution in Table 4, it appears to the writer that the author's method is encircled so that it has no useful field.

| | 0.375 *AD* | 0.625 *AB* | 0.36 *BA* | 0.28 *BE* | 0.36 *BC* | | 0.625 *CB* | 0.375 *CF* | |
|---|---|---|---|---|---|---|---|---|---|
| | .... | −84 | +28 | .... | .... | | .... | .... | |
| | +31½ | +52½ | −10 | − 8 | −10 | | .... | .... | |
| | .... | − 5 | +26 | +11 | .... | | − 5 | − 6 | |
| | + 2 | + 3 | −13½ | −10½ | −13 | | + 7 | + 4 | |
| | − 4 | − 7 | + 1½ | + 1½ | + 3½ | | − 7 | − 2 | |
| | + 4 | + 7 | − 2¼ | − 2 | − 2¼ | | + 5½ | + 3½ | |
| | − ½ | − 1 | + 3½ | + 1½ | + 3 | | − 1 | − ½ | |
| | + ½ | + 1 | − 3 | − 2 | − 3 | | + 1 | + ½ | |
| | +33½ | −33½ | +30½ | − 8½ | −22 | | + ½ | − ½ | |
| 0.31 *DG* | 0.23 *DA* | 0.46 *DE* | 0.285 *ED* | 0.19 *EB* | 0.285 *EF* | 0.24 *EH* | 0.46 *FE* | 0.23 *FC* | 0.31 *FK* |
| .... | .... | .... | .... | .... | −115 | .... | +52 | .... | .... |
| .... | .... | .... | +32½ | +22½ | + 32½ | +27½ | −24 | −12 | −16 |
| .... | +16 | +16 | .... | − 4 | − 12 | .... | +16 | .... | .... |
| −10 | − 7 | −15 | + 4½ | + 3 | + 4½ | + 4 | − 7 | − 4 | − 5 |
| .... | + 1 | + 2 | − 7½ | − 5 | − 3½ | .... | + 2 | + 2 | .... |
| − 1 | − 1 | − 1 | + 4½ | + 3 | + 4½ | + 4 | − 2 | − 1 | − 1 |
| .... | + 2 | + 2 | − ½ | − 1 | − 1 | .... | + 2 | + 2 | .... |
| − 1 | − 1 | − 2 | + 1 | + ¼ | + 1 | + ¼ | − 2 | − 1 | − 1 |
| −12 | +10 | + 2 | +34½ | +18½ | − 89 | +36 | +37 | −14 | −23 |

FIG. 10.—FIFTEEN-MINUTE SOLUTION BY L. BLUME

It also may be mentioned that for one-story structures the "quality" type of solution will be considerably faster than either end-moment distribution or the author's method.

Correction for *Transactions:* In May, 1942, *Proceedings*, page 684, Eq. 3, change "$\Sigma M'$" to "$-\Sigma M'$."

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DISCUSSIONS

# PROFILE CURVES FOR OPEN-CHANNEL FLOW

## Discussion

BY C. J. POSEY, ASSOC. M. AM. SOC. C. E.

C. J. POSEY,[7] ASSOC. M. AM. SOC. C. E.[7a]—Upon reading this paper, one wavers between two conclusions—that Professor Gunder is wrong, or that the backwater curves shown in standard texts are seriously in error. Fortunately, neither conclusion is correct. The principal difference lies in the shape of the type 3 curves in channels of mild, zero, adverse, or critical slope. According to Fig. 1, the $M_3$, $O_3$, and $A_3$ curves start perpendicular to the bottom of the channel and are concave downward for some distance before becoming concave upward. Previously published sketches of these curves have shown them as concave upward from end to end. The $C_3$ curve in Fig. 1 starts upward perpendicular to the bottom and is concave downward all the way to the critical depth line. Heretofore it has been shown as a simple horizontal line.

Actually, the discrepancies are not as serious as one might infer from examination of Fig. 1. The author himself states that the point of inflection lies near the bottom of the channel—very close to it if the Bazin or Ganguillet-Kutter expressions are used for the Chézy coefficient. Even if the Manning relationship is used the discrepancies are not important. In drawing small-scale illustrations such as Fig. 1, the true shape of the curves often must be modified. That Fig. 1 has been somewhat distorted is shown by the fact that the curves start up from the bottom at right angles to it, whereas, according to the text of the paper, they should start up vertically. This peculiar feature of the curves, which is the most surprising result of Professor Gunder's studies, and which makes the reversal of curvature necessary, can easily be shown to be consistent with the assumptions ordinarily made in computing backwater curves.

The rate of loss of head through friction, with respect to distance along the channel, is assumed to be given by Chézy's formula

$$S_f = \frac{Q^2}{C^2 y^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

NOTE.—This paper by Dwight F. Gunder, Esq., was published in April, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: June, 1942, by Carl Rohwer, Assoc. M. Am. Soc. C. E.

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[7a] Received by the Secretary August 3, 1942.

In the range of infinitesimal depths close to the bottom of the channel the velocity head is very great, and changes at a rapid rate, whereas the elevation head is scarcely changing at all. In a rectangular channel the rate of change of velocity head, with respect to distance along the channel, is given by

$$\frac{d}{dx}\left(\frac{V^2}{2g}\right) = \frac{d}{dx}\left(\frac{Q^2}{2g\,y^2}\right) = -\frac{Q^2}{g\,y^3}\frac{dy}{dx} \quad \ldots\ldots\ldots\ldots (21)$$

Since the change of elevation head is comparatively insignificant, the friction loss is derived almost entirely from the change of velocity head. Equating the values given by Eqs. 20 and 21, and changing signs to allow for the implicit negative sign of $S_f$, one finds that for depths close to zero

$$\frac{dy}{dx} = \frac{g}{C^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

From this expression it is evident that if any formula is used in which $C$ approaches zero as $y$ approaches zero, the corresponding water-surface slope will be infinite. The limit is entirely theoretical, of course, representing a flow condition impossible of attainment.

The evaluation of friction in the region near this limit, where the velocity of flow is many times the Belanger critical, is very uncertain. It is customary to assume that the rate of loss of head by friction, in variable flow, is the same as in uniform flow at the same depth and velocity. The rate of loss in uniform flow is then computed by one of the standard open-channel formulas. It is doubtful, however, whether any open-channel formula has a proven range extending to velocities greater than twice the critical. Observations of extremely high velocities are difficult, and flow on slopes steep enough to sustain such velocities is almost certain to become pulsating or "slug" flow rather than uniform flow.[8] Although uniform flow at the high velocities existing in the lower portions of the type 3 curves is nearly impossible, variable flow in this range can exist. Examples of computed profiles for a common case are shown in Fig. 2. A discharge of 507 cu ft per sec flows out from under a gate into a channel with horizontal bottom. Flow is from left to right in the figure, and two possible conditions of flow starting from different gate heights are shown. The Manning formula is used, with $n$ taken as 0.016. If the bottom of the gate is 10 ft above the horizontal bottom of the channel, the water is backed up to a depth of 50 ft behind the gate. If the bottom of the gate is 4 ft above the bottom of the channel, the depth upstream becomes 254 ft. In either case the water-surface profile computed by Eq. 19 follows the same curve, crossing the critical depth vertically at a point 3,370 ft downstream from the gate when the opening is 10 ft, or 6,250 ft downstream when the opening is 4 ft.

Water-surface profiles computed using the constant value of $C$ which corresponds to the median value of $y$ are also shown in Fig. 2. The curve downstream from the 10-ft gate reaches the critical depth in a distance of 3,500 ft, 4% in excess of the value given by Eq. 19. The curve starting at the 4-ft gate reaches

[8] "A Criterion for Instability of Flow in Steep Channels," by G. H. Keulegan and G. W. Patterson, *Transactions*, Am. Geophysical Union, 1940, Pt. II, p. 594.

the critical depth 6,760 ft downstream, a distance 7½% greater than that given by Eq. 19. Although it is evident that the discrepancy would be greater for smaller gate openings, the 4-ft gate opening is already beyond the range of values likely to be encountered in practice. The discrepancies could be de-

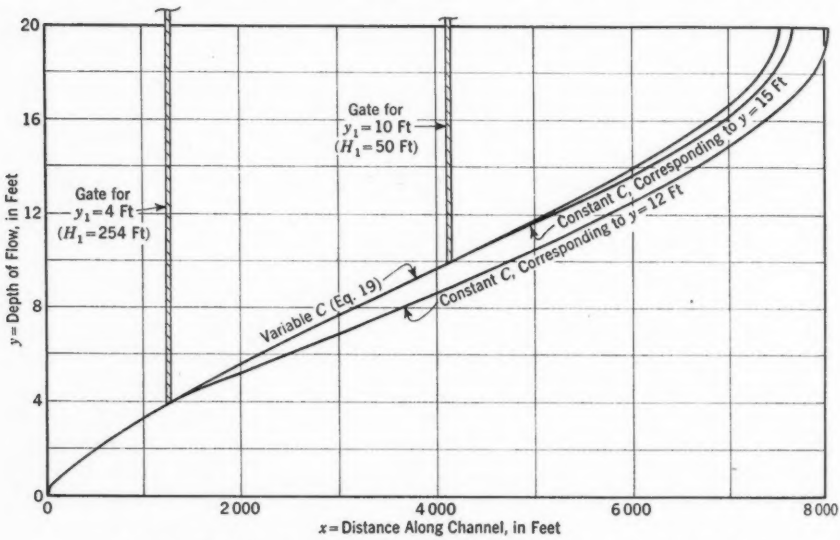


FIG. 2.—WATER-SURFACE PROFILES OF THE $O_3$ TYPE, BASED ON VARIABLE OR CONSTANT CHÉZY COEFFICIENTS

creased by using a value of $C$ corresponding to an estimated average value of $y$ rather than the median value. For example, with the gate at 10 ft, a value of $C$ corresponding to a depth of 13 ft gives a result differing from that computed by Eq. 19 by less than 1%.

With regard to simplicity, it is not much easier to use a constant value of $C$ in computing the $O_3$ curves than to use Eq. 19, but the use of a constant value of $C$ in computing $M_3$ and $A_3$ curves permits the use of Bresse's function,[9] with great saving in labor over that required in using Eqs. 17 and 18.

From a practical viewpoint, there seems to be little choice between results obtained using a variable Chézy coefficient and those obtained using a constant coefficient. The effect of channel shape will ordinarily be of greater importance. Until a comparison has been made with data from variable high-velocity flow in wide rectangular channels, neither method can be considered to have superior accuracy. The author's prediction of a reversal of curvature in the type 3 curves, a phenomenon which may be observable in channels of any shape, should help to stimulate interest in obtaining actual measurements.

[9] Tables of Bresse's function for adverse as well as mild and steep slopes are given in "Hydraulics of Steady Flow in Open Channels," by Sherman M. Woodward and Chesley J. Posey, John Wiley & Sons, New York, N. Y., 1941.

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DISCUSSIONS

# NUMERICAL PROCEDURE FOR COMPUTING DEFLECTIONS, MOMENTS, AND BUCKLING LOADS

## Discussion

BY MESSRS. JOHN B. WILBUR, RALPH W. STEWART, AND STEFAN J. FRAENKEL

JOHN B. WILBUR,[21] ASSOC. M. AM. SOC. C. E.[21a]—Professor Newmark's paper has been stimulating to the writer, particularly since it places emphasis on mathematically simple approaches to problems in elastic stability. Although Professor Newmark shows that simultaneous equations may be used to insure that the assumed deflection curve of a compression strut is essentially proportional to its shape as computed elastically on the basis of the assumed deflection curve, his detailed treatment is devoted to the substitution of a method of successive approximations for the solution of simultaneous equations.

The writer solved the problem illustrated by Fig. 13 on the basis of five simultaneous equations, and obtained substantially the same results. This solution was relatively straightforward, although the form of the simultaneous equations was such that it was convenient to adopt a method of successive approximations in their solution. Having successfully completed this solution, it was then decided to investigate the accuracy of simpler direct solutions, based on two simultaneous equations only. Symmetrical cases only were considered; otherwise three equations would have been necessary. The results of these relatively simple solutions were quite satisfactory, as is illustrated in the following discussion.

Consider first the case of Fig. 13. Assuming that the deflections of the strut at the center and quarter points are $\delta$ and $a\,\delta$, respectively, the equivalent concentrated $\frac{M}{EI}$-loads at the center and quarter points are computed from the relations for parabolic loading given in Fig. 5(*b*). These loads, together with the resultant net reaction, are shown in Fig. 24. By the moment area

NOTE.—This paper by N. M. Newmark, Assoc. M. Am. Soc. C. E., was published in May, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1942, by Bruce Johnston, Assoc. M. Am. Soc. C. E., and June, 1942, by M. S. Ketchum, Jr., Assoc. M. Am. Soc. C. E.

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[21a] Received by the Secretary June 4, 1942.

method:

$$a\,\delta = \frac{P\,L\,\delta}{48\,E\,I}(11\,a + 6)\,\frac{L}{4} = \frac{P\,L^2\,\delta}{192\,E\,I}(11\,a + 6) \quad \ldots\ldots (21a)$$

and

$$\delta = \frac{P\,L\,\delta}{48\,E\,I}(11\,a + 6)\,\frac{2\,L}{4} - \frac{P\,L\,\delta}{48\,E\,I}(10\,a + 1)\,\frac{L}{4} = \frac{P\,L^2\,\delta}{192\,E\,I}(12\,a + 11) \quad (21b)$$

Let $A = \dfrac{P\,L^2}{192\,E\,I}$ and Eqs. 21 become

$$a = A\,(11\,a + 6) \quad \ldots\ldots (22a)$$

and

$$1 = A\,(12\,a + 11) \quad \ldots\ldots (22b)$$

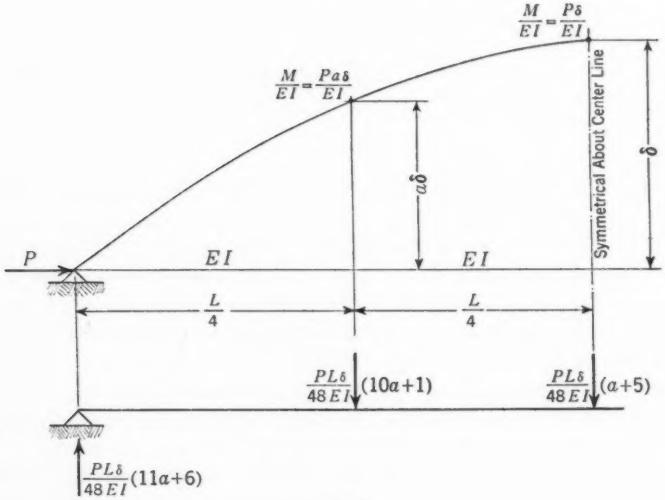



FIG. 24

Solving simultaneously: $a = 0.707$; $A = \dfrac{1}{19.48} = \dfrac{P\,L^2}{192\,E\,I}$. Hence, $P_{cr} = \dfrac{192}{19.48}\,\dfrac{E\,I}{L^2} = \dfrac{9.86\,E\,I}{L^2}$ vs. $\dfrac{\pi^2\,E\,I}{L^2}$ (exactly).

Before applying this procedure to the case illustrated by Professor Newmark in Fig. 15, where $I$ is not constant throughout the length of the strut, it was convenient to develop equations for equivalent concentrated loads for parabolic loading curves for the case in which the spans of the two adjacent segments are not equal. Referring to Fig. 25, these equations are as follows:

$$R_{bc} = \frac{l_2}{12\,l_1\,(l_1 + l_2)}\left[a\,(-\,l_2^2) + b\,(4\,l_1 + l_2)(l_1 + l_2) + c\,l_1\,(2\,l_1 + l_2)\right] \ldots (23a)$$

and

$$R_{ba} = \frac{l_1}{12\,l_2\,(l_1 + l_2)}\left[a\,l_2\,(l_1 + 2\,l_2) + b\,(l_1 + 4\,l_2)(l_1 + l_2) + c\,(-\,l_1^2)\right] \ldots (23b)$$

Assuming that the deflections of the strut at the center and the 0.2-points are $\delta$ and $a\,\delta$, respectively, the equivalent concentrated $\frac{M}{E\,I}$-load at the 0.2-point is computed from the foregoing relations for parabolic loadings, whereas

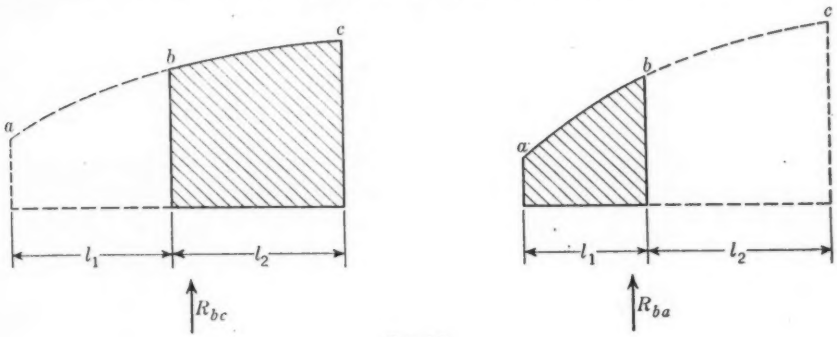



FIG. 25

the equivalent concentrated $\frac{M}{E\,I}$-load at the center is computed on the basis of parabolic loadings of equal adjacent sections. These loads, together with the

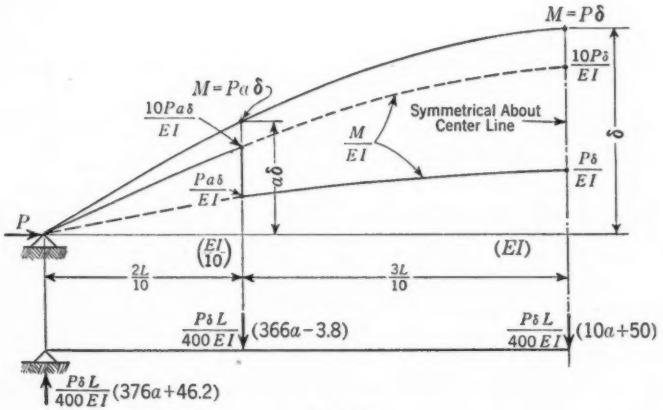



FIG. 26

resultant net reaction, are shown in Fig. 26. By the moment-area method,

$$a\,\delta = \frac{P\,\delta\,L}{400\,E\,I}(376\,a + 46.2)\frac{2\,L}{10} = \frac{P\,L^2\,\delta}{4{,}000\,E\,I}(752\,a + 92.4)\ldots(24a)$$

and

$$\delta = \frac{P\,\delta\,L}{400\,E\,I}(376\,a + 46.2)\frac{5\,L}{10} - \frac{P\,\delta\,L}{400\,E\,I}(366\,a - 3.8)\frac{3\,L}{10}$$

$$= \frac{P\,L^2\,\delta}{4{,}000\,E\,I}(782\,a + 219.6)\ldots\ldots\ldots\ldots\ldots\ldots(24b)$$

Let $A = \frac{P L^2}{4{,}000\, E\, I}$, and Eqs. 24 become

$$a = A\,(752\,a + 92.4) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25a)$$

and

$$1 = A\,(782\,a + 219.6) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25b)$$

Solving simultaneously: $a = 0.823$; $A = \frac{1}{863} = \frac{P L^2}{4{,}000\, E\, I}$. Hence, $P_{cr} = \frac{4{,}000}{863}\frac{L^2}{E\, I} = 4.64\,\frac{L^2}{E\, I}$ vs. $\frac{4.5\, L^2}{E\, I}$ (exactly).

Since, for a strut composed of sections of constant $E\, I$, the deflection curve must be compounded of sine curves, an interesting exact solution to the foregoing problem may be made as follows, with reference to Fig. 27.

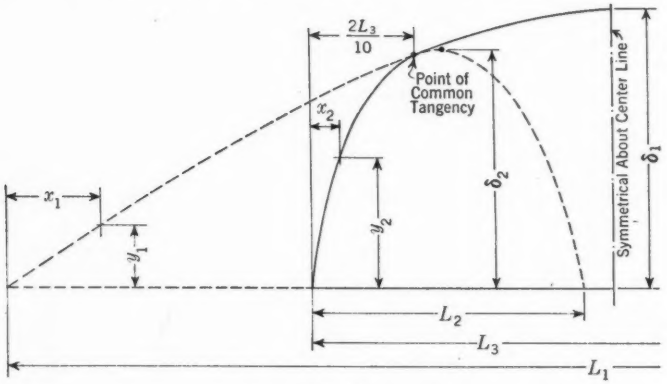



FIG. 27

For the case under consideration, the true deflection curve, shown by the full line, is composed of segments of the curves

$$y_1 = \delta_1 \sin \frac{\pi\, x_1}{L_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26a)$$

and

$$y_2 = \delta_2 \sin \frac{\pi\, x_2}{L_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26b)$$

At the point of common tangency, where

$$x_1 = \frac{L_1 - L_3}{2} + \frac{2\, L_3}{10} = \frac{1}{10}\,(5\, L_1 - 3\, L_3) \ldots\ldots\ldots\ldots (27a)$$

and

$$x_2 = \frac{L_3}{5} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27b)$$

the following relations hold:

$$y_1 = y_2;\ \frac{dy_1}{dx_1} = \frac{dy_2}{dx_2};\ \text{and}\ \frac{d^2y_1}{dx^2_1} = \frac{1}{10}\frac{d^2y_2}{dx^2_2},\ \text{since}\ I_1 = 10\, I_2.$$

These relations reduce to

$$\delta_1 \sin \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \delta_2 \sin \frac{\pi}{5} \frac{L_3}{L_2} \ldots\ldots\ldots\ldots (28a)$$

$$\frac{\delta_1}{L_1} \cos \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \frac{\delta_2}{L_2} \cos \frac{\pi}{5} \frac{L_3}{L_2} \ldots\ldots\ldots\ldots (28b)$$

and

$$\frac{\delta_1}{L^2_1} \sin \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \frac{\delta_2}{10\,L^2_2} \sin \frac{\pi}{5} \frac{L_3}{L_2} \ldots\ldots\ldots\ldots (28c)$$

Dividing Eq. 28*c* by Eq. 28*a*, $\frac{1}{L^2_1} = \frac{1}{10\,L^2_2}$; whence $L_2 = \frac{L_1}{\sqrt{10}}$. Eqs. 28*a* and 28*b* become

$$\delta_1 \sin \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \delta_2 \sin \frac{\pi \sqrt{10}}{5} \frac{L_3}{L_1} \ldots\ldots\ldots\ldots (29a)$$

and

$$\delta_1 \cos \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \delta_2 \sqrt{10} \cos \frac{\pi \sqrt{10}}{5} \frac{L_3}{L_1} \ldots\ldots\ldots\ldots (29b)$$

Dividing Eq. 29*a* by Eq. 29*b*,

$$\sqrt{10} \tan \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \tan \frac{\pi \sqrt{10}}{5} \frac{L_3}{L_1} \ldots\ldots\ldots\ldots (30)$$

and $\frac{L_3}{L_1} = 0.675$. Hence, $L_1 = 1.481\,L_3$. This gives the "reduced" length of the strut under consideration. The critical load is then given by

$$P_{cr} = \frac{\pi^2\,E\,I}{(1.481\,L_3)^2} = \frac{4.50\,E\,I}{L^2_3} \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

In the opinion of the writer, Professor Newmark has performed a definite service since he has dissociated one type of problem in elastic stability from the shroud of the formal solution of differential equations. It is to be hoped that relatively simple methods of solution will eventually be developed for other types of problems in elastic stability.

RALPH W. STEWART,[22] M. AM. SOC. C. E.[22a]—Effective procedures for determining the elastic curves of beams are presented in compact form by the author. The paper has value in an engineer's reference library since the analyses demonstrated have heretofore been scattered in different treatises. The use of progressive load, shear, and moment increments to establish deflections and the alinement of elastic curves is familiar in the analysis of arches. An excellent demonstration of the use of the device termed "linear correction to moments" (which, by analogy, is the same as "linear correction to deflections") has been presented by A. W. Buel, M. Am. Soc. C. E., and C. S. Hill.[23]

[22] Engr. of Bridge and Structural Design, City of Los Angeles, Los Angeles, Calif.

[22a] Received by the Secretary July 6, 1942.

[23] "Reinforced Concrete," by A. W. Buel and C. S. Hill, The Engineering News Publishing Co., New York, N. Y., 1906, Fig. 43, p. 140.

It appears to the writer that a part of the paper dealing with beams of variable section can be improved in analytical procedure and also that a rather important deficiency in the author's illustration of the use of his computations justifies a rewriting of this part of the paper.

To clarify and justify this opinion, a statement regarding the basic constants of beam flexure is necessary. A basic constant is defined as being either a simple constant quantity or a constant ratio between two variable quantities. A constant that is more complex than a basic constant will be referred to as a derived constant because it is derived by the use of two or more basic constants.

The basic constants of beam flexure are well illustrated by the dimensions governing railway curves. A circular railway curve is completely determined by the length of a tangent and the angle of intersection of the tangents. The transition spiral often used at the end of a circular curve is fully determined by the length of its tangents and their angle of intersection. If the over-all length of the spiral curve is known (as the span of a beam is known), then one tangent and the angle of intersection are sufficient. The basic constants governing the flexure of a beam that has no more than two supports are similar and are as follows, the distance between supports being known:

(1) The basic stiffness of the left end of the beam. This is the ratio of the moment to the angle of intersection between the tangents to the elastic curve when the beam is hinged at the right end and a moment is applied at the left end.

(2) The ratio of the length of one tangent to the elastic curve to the length of the beam for the same condition of flexure as in (1).

(3) and (4) These are the same as (1) and (2), except that the left end of the beam is taken as hinged and the moment is applied at the right end.

(5) The angle of intersection between the tangents of the elastic curve when the beam acts as a simply supported beam subjected to its loads.

(6) The ratio of the length of one tangent of the elastic curve to the length of the beam for the simply supported condition.

The first four constants are "beam constants," which are independent of the loading. The last two are "load constants," which depend on the loads.

Of the six basic constants of flexure, only five are independent, since the principle of Maxwell's theorem of reciprocal deflections (angular) will enable any missing beam constant to be computed from the others. It is understood in beam flexure that the tangents may be taken as equal to their projected length in the unsprung beam.

An end slope is not a basic flexure constant. This can be seen from the fact that in a cantilever beam the slope at the fixed end is known to be zero; but this knowledge, combined with the length of a tangent to the elastic curve, is insufficient to determine the curve. If, however, the length of a tangent and the intersection angle between tangents are known, the curve is determined.

The Hardy Cross stiffness factor is not a basic constant because it is a function of an end slope and also two basic stiffnesses, one at each end of the beam.

Much valuable time has been lost and is still being lost by engineering designers through injudicious selection of flexure constants in analyzing the various types of structures. The author's treatment of the beam in Figs. 10 and 11 would tend to encourage rather than correct this tendency. A revised

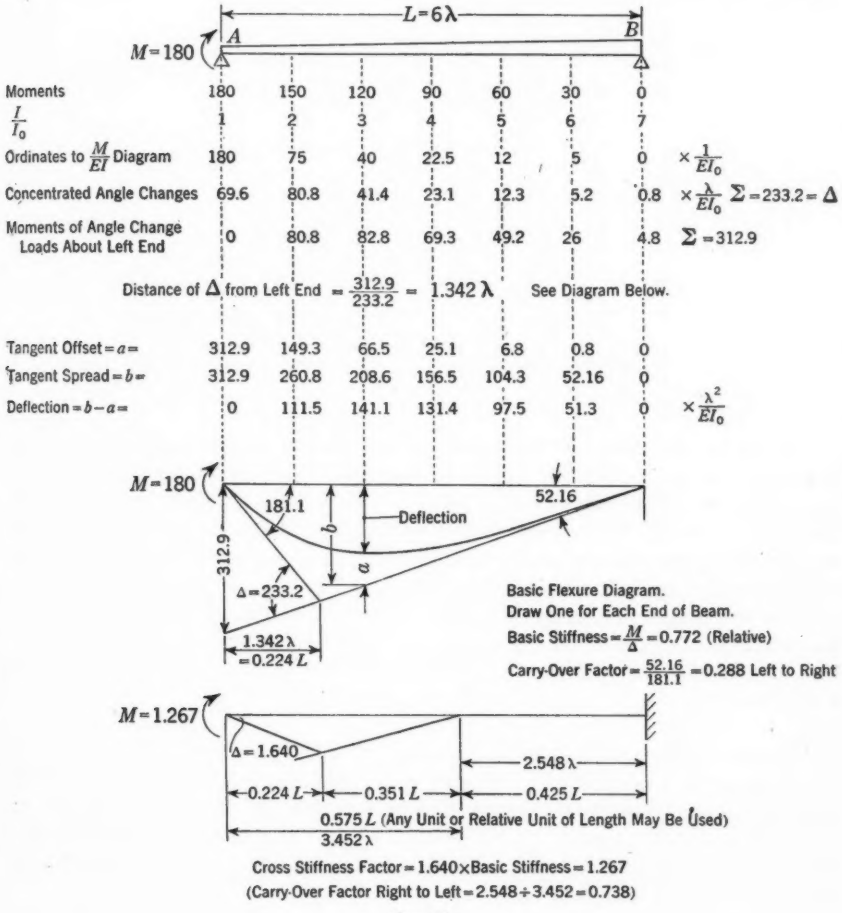

L=6λ
M=180
A
B
Moments 180 150 120 90 60 30 0
I/I0 1 2 3 4 5 6 7
Ordinates to M/EI Diagram 180 75 40 22.5 12 5 0 × 1/EI0
Concentrated Angle Changes 69.6 80.8 41.4 23.1 12.3 5.2 0.8 × λ/EI0 Σ=233.2=Δ
Moments of Angle Change Loads About Left End 0 80.8 82.8 69.3 49.2 26 4.8 Σ=312.9
Distance of Δ from Left End = 312.9/233.2 = 1.342 λ See Diagram Below.
Tangent Offset=a= 312.9 149.3 66.5 25.1 6.8 0.8 0
Tangent Spread=b= 312.9 260.8 208.6 156.5 104.3 52.16 0
Deflection=b−a= 0 111.5 141.1 131.4 97.5 51.3 0 × λ²/EI0
M=180
181.1
52.16
Deflection
b
a
312.9
Δ=233.2
1.342λ
=0.224 L
Basic Flexure Diagram.
Draw One for Each End of Beam.
Basic Stiffness = M/Δ = 0.772 (Relative)
Carry-Over Factor = 52.16/181.1 = 0.288 Left to Right
M=1.267
Δ=1.640
2.548λ
0.224 L
0.351 L
0.425 L
0.575 L (Any Unit or Relative Unit of Length May Be Used)
3.452λ
Cross Stiffness Factor = 1.640 × Basic Stiffness = 1.267
(Carry-Over Factor Right to Left = 2.548 ÷ 3.452 = 0.738)

FIG. 28

computation of the properties of this beam is therefore offered in Fig. 28, and the justification for this revised computation is illustrated by Fig. 29. The columns in Fig. 29 are the same members whose properties are determined in Fig. 28. The deck is of constant section with relative stiffness as shown. The sidesway moments and the appurtenant lateral force as shown in Fig. 29 are computed by the following consecutive steps:

Write in deck traverse angles 1, 2, and 3. Multiply stiffness, 2.97, by angle value, 3, to get moment 8.91; divide moment 8.91 by stiffness 1.485 to

get column traverse angle 6. Add angles 1 and 6 to get bottom column traverse angle 7; multiply angle 7 by stiffness 0.772 to get moment 5.404. Add all column moments and divide by frame height to get lateral force. The effect of any other lateral force will be in direct proportion. The lateral deflection of the structure follows as a direct by-product.

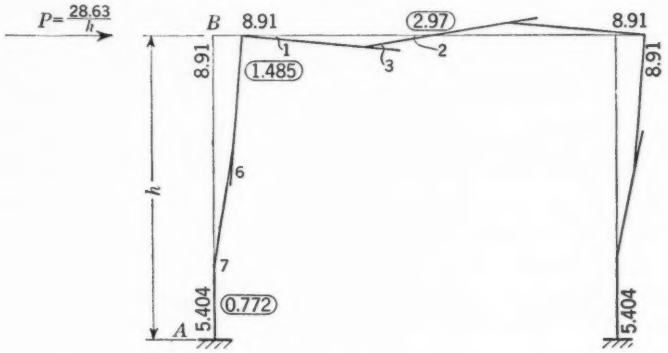



FIG. 29

It can be seen that the foregoing computation of sidesway moments, using basic constants, can be done in less than one minute, whereas computing these moments, with the derived constants, which are the only ones mentioned by the author, would take much longer, would give a less accurate result, and would fail to give the deflection of the structure as a direct by-product. The same situation would prevail with vertical loads.

The usefulness of the computation of deflections and elastic properties of an individual beam is obviously increased by bringing the results into a form that can be used for the quick computation of the deflections and moments of a structure in which the beam becomes a member.

STEFAN J. FRAENKEL,[24] JUN. AM. SOC. C. E.[24a]—Recently the writer was called upon to investigate the stresses in a 125-ft derrick boom. Since such long and slender members are subjected to considerable deformation which, in turn, influences the stresses, it was necessary to determine the deflection of this boom. The methods outlined in Professor Newmark's able paper lent themselves well to this investigation.

The boom (see Fig. 30) consisted of four angles—two 6-in. by 4-in. by $\frac{1}{2}$-in. angles, which formed the top chord, and two 4-in. by 4-in. by $\frac{1}{2}$-in. angles, which constituted the bottom chord. A preliminary investigation indicated that the strengthening of the middle section (which was 59 ft long) with 3-in. by $\frac{1}{2}$-in. bars at the top and 3-in. by $\frac{3}{8}$-in. bars at the bottom was advisable. That is the section shown in Fig. 30, which was also used in the investigation related herein. The section of the boom varied considerably outside the middle part, as is shown by the variations in the values of $I$ and $y$.

[24] With Eng. Dept., Pittsburgh-Des Moines Steel Co., Des Moines, Iowa.

[24a] Received by the Secretary July 8, 1942

A general view of the boom is shown in Fig. 31. The boom will be analyzed in a horizontal position. Because of the fact that its cross section is unsymmetrical, the neutral axis does not coincide with the geometrical center line,

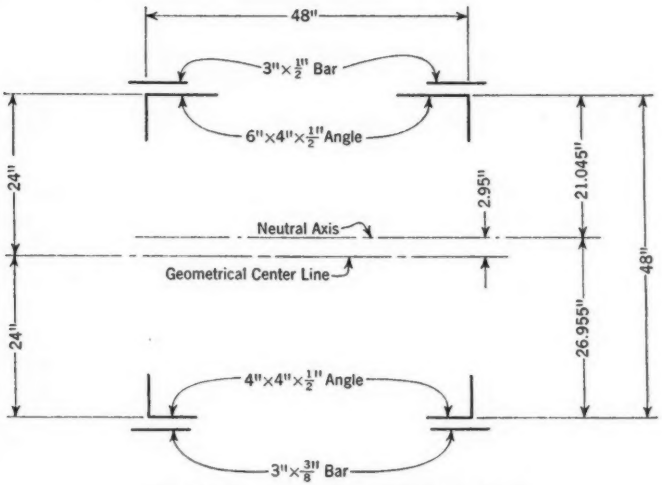



FIG. 30.—SECTION IN CENTER PART OF BOOM

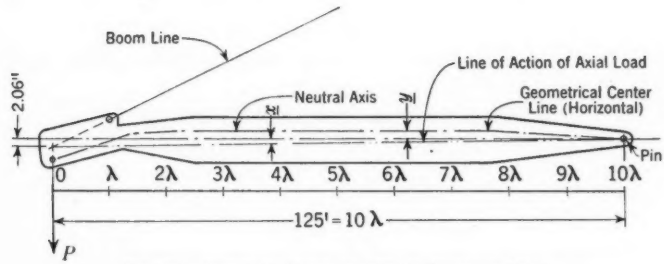



FIG. 31.—GENERAL VIEW OF 125-FT DERRICK BOOM

the distance between them at any point being denoted by $y$. The line of action of the axial loads is determined at the right end of the boom by the location of the pin, and at the left end by the intersection of the boom line and

TABLE 2.—VALUES OF $x$, $y$, AND $I$ FOR VARIOUS VALUES OF λ

| Symbol | 0 | 1λ | 2λ | 3λ | 4λ | 5λ | 6λ | 7λ | 8λ | 9λ | 10λ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 2.06 | 1.86 | 1.65 | 1.45 | 1.24 | 1.03 | 0.83 | 0.62 | 0.41 | 0.21 | 0.00 |
| $y$ | −8.39 | 1.13 | 2.27 | 2.95 | 2.95 | 2.95 | 2.95 | 2.95 | 2.27 | 1.76 | 0.00 |
| $I$ | 42,057 | 5,363 | 7,098 | 11,843 | 11,843 | 11,843 | 11,843 | 11,843 | 7,098 | 3,387 | 5,071 |
| $I$ | 12.4 $I_o$ | 1.59 $I_o$ | 2.09 $I_o$ | 3.49 $I_o$ | 3.49 $I_o$ | 3.49 $I_o$ | 3.49 $I_o$ | 3.49 $I_o$ | 2.09 $I_o$ | 1.00 $I_o$ | 1.50 $I_o$ |

the vertical line of action of the load $P$. (It should be understood that $P$ includes both live load and the left reaction of the boom due to its own weight.) This intersection (which, of course, is imaginary) occurs for the case of the

flat boom 2.06 in. below the geometrical center line. The distance at any point between the geometrical center line and the line of action of the axial load will be denoted by $x$. A summary of the values of $x$, $y$, and $I$ for various values of $\lambda$ is given in Table 2.

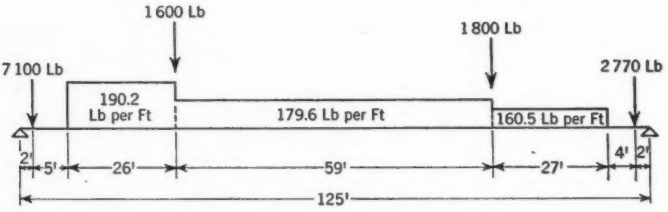



FIG. 32.—LOADING SKETCH OF BOOM

Fig. 32 shows the loading of the boom due to its own weight. The dead-load reaction at the left end causes an axial load of 51,250 lb, and the maximum axial load due to live load is 64,950 lb. In computing the deflections the following conventions were adopted:

(1) Downward deflection is positive;
(2) Inch and pound units are employed in the computations;
(3) The modulus of elasticity, $E$, is taken as 30 by $10^6$ lb per sq in.; and
(4) The boom is divided into ten parts, so that $10\lambda = 125$ ft.

TABLE 3.—DETERMINATION OF DOWNWARD DEFLECTION

| Line | Description | Tenth Points: 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| | Moment of inertia I (in.⁴) | 42,057 | 5,363 | 7,098 | 11,834 |
| 1 | Moment due to weight of boom | 0 | 1.888 | 3.313 | 4.313 |
| 2 | Distributed angle change | 0 | 351 | 469 | 365 |
| 3 | Concentrated angle change | | 312.2 | 432.0 | 388.8 |
| 4 | Assumed average slope | | 1,600 | 1,287.8 | 855.8 467.0 |
| 5 | Trial deflection | | 1,600 | 2,887.8 | 3,743.6 |
| 6 | Linear correction to deflection | | +147.7 | 295.4 | 443.1 |
| 7 | Resultant deflection | 0 | 1,747.7 | 3,183.2 | 4,186.7 |
| 8 | Resultant deflection in inches = $\delta_w$ | 0 | 1.31 | 2.39 | 3.14 |

The total deflection, $\Delta$, was considered to be made up of three constituent parts:

(*a*) $\delta_w$ = deflection caused by bending moment due to the weight of the boom;
(*b*) $\delta_a$ = deflection due to axial dead load; and
(*c*) $\Delta_L$ = deflection due to axial live load.

The following relations exist between these deflections:

$$\delta_w + \delta_a = \Delta_D \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32a)$$

and

$$\Delta_D + \Delta_L = \Delta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32b)$$

*Determination of* $\delta_w$.—The necessary computations are recorded in Table 3, and no further explanations are required. Of course, it would not have been necessary to compute the equivalent concentrated angle changes, since there were no discontinuities in the $\frac{M}{E\,I}$-diagram, and the simplified method mentioned in Part II of the paper could have been used.

*Determination of* $\delta_a$ *and* $\Delta_D$.—This component of the deflection is the upward movement caused by the moment that the axial dead load has about the neutral axis. That this movement is upward can be ascertained from the fact that, except at the left end, the line of action of the axial load is below the neutral axis. The moment arm is $-(x + y + \delta_a)$ and is independent of the deflection $\delta_w$. This can be justified as follows: Assume the boom to be fully supported throughout its length so that there is no deflection. Then remove the supports except for the reactions, and the boom will deflect under the simultaneous action of bending moment due to weight and axial load due to weight. The moment arm of the axial load will be the distance between the line of action of the axial load and the deflected neutral axis—namely, $-(x + y + \delta_a)$, at the time the intermediate supports are removed, and therefore will not depend upon the deflection $\delta_w$. The moment arm is given a minus sign inasmuch as the moment will cause an upward, or negative, deflection. All computations for $\delta_a$ and the values of $\Delta_D$ are given in Table 4.

CAUSED BY BENDING MOMENT DUE TO WEIGHT OF BOOM ($\delta_w$)

| Tenth Points: 4 | 5 | 6 | 7 | 8 | 9 | 10 | Common factor |
|---|---|---|---|---|---|---|---|
| 11,834 | 11,834 | 11,834 | 11,834 | 7,098 | 3,387 | 5,071 | .... |
| | | | $10\lambda$ = 125 Ft | | | | |
| 4.777 | 4.925 | 4.741 | 4.237 | 3.188 | 1.760 | 0 | .... |
| 404 | 416 | 400 | 358 | 449 | 520 | 0 | $10^6$ |
| 399.5 | 411.3 | 395.7 | 380.2 | 445.7 | 421.5 | | $1/E$ |
| 67.5 | −343.8 | −739.5 | −1,119.7 | −1,565.4 | −1,986.9 | | $\lambda/E$ |
| 4,210.6 | 4,278.1 | 3,934.3 | 3,194.8 | 2,075.1 | 509.7 | −1,477.2 | $\lambda/E$ |
| 590.8 | 738.5 | 886.2 | 1,033.9 | 1,181.6 | 1,329.3 | +1,477.2 | $\lambda^2/E$ |
| 4,801.4 | 5,016.6 | 4,820.5 | 4,228.7 | 3,256.7 | 1,839.0 | 0 | $\lambda^2/E$ |
| 3.60 | 3.77 | 3.62 | 3.17 | 2.44 | 1.38 | 0 | $\lambda^2/E$ |

*Determination of* $\Delta_L$ *and* $\Delta$.—The axial live load also will cause an upward deflection, since its line of action lies below the deflected neutral axis. The moment arm of the axial live load, however, will be much smaller since the boom deflects by the amount $\Delta_D$ before the live load is applied. The arm is $-(x + y + \Delta_L - \Delta_D)$, the minus sign being given because of the resulting upward deflection. It is not possible to use Eq. 4 or 6 in estimating the

TABLE 4.—DETERMINATION OF UPWARD DEFLECTION DUE TO

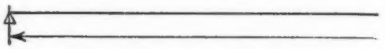
| Line | Description | Tenth Points: 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| | Moment of inertia I (in.⁴) | 42,057 | 5,363 | 7,098 | 11,834 |
| | |  | | | |
| 1 | Assumed deflection due to axial dead load $\delta_a$ | 0 | −0.06 | −0.11 | −0.15 |
| 2 | Moment arm $(-x-y-\delta_a)$ | +6.33 | −3.05 | −4.03 | −4.55 |
| 3 | Moment (51,250 times moment arm) | +324.0 | −156.3 | −206.5 | −233.4 |
| 4 | Distributed angle change | −7.71 | +29.10 | +29.10 | +19.68 |
| 5 | Assumed average slope | −80.00 | −50.90 | −21.80 | −2.12 |
| 6 | Trial deflection (1) | −223.24 | −303.24 | −354.14 | −375.94 |
| 7 | Trial deflection (2) | −0.64 | +2.42 | +2.42 | +1.64 |
| 8 | Linear correction to deflection | +223.88 | +201.51 | +179.12 | +156.73 |
| 9 | Resultant deflection $\delta'_a$ | 0 | −99.31 | −172.60 | 217.57 |
| 10 | Resultant deflection $\delta'_a$ | 0 | 0.075 | 0.130 | 0.164 |
| | | | | | Final Deflections |
| 11 | .... | 0 | −0.075 | −0.130 | −0.164 |
| | | | | | Total Dead Load |
| 12 | .... | 0 | 1.24 | 2.26 | 2.98 |

TABLE 5.—DETERMINATION OF UPWARD DEFLECTION

| Line | Description | Tenth Points: 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| | Moment of inertia I (in.⁴) | 42,057 | 5,363 | 7,098 | 11,834 |
| | | | | | |
| 1 | Assumed deflection due to axial live load $\Delta_L$ | 0 | −0.08 | −0.078 | −0.067 |
| 2 | Moment arm $(-x-y+\Delta_D-\Delta_L)$ | +6.33 | −1.79 | −1.738 | −1.487 |
| 3 | Moment (64,950 times moment arm) | 411.0 | −116.2 | −112.9 | −96.5 |
| 4 | Distributed angle change | −9.78 | +21.65 | +15.90 | +8.15 |
| 5 | Assumed average slope | −50.00 | −28.35 | −12.45 | −4.30 |
| 6 | Trial deflection (1) | +45.56 | −4.44 | −32.79 | −45.24 |
| 7 | Trial deflection (2) | −0.81 | +1.81 | +1.33 | +0.68 |
| 8 | Linear correction to deflection | −44.75 | −40.27 | −35.80 | −31.32 |
| 9 | Resultant deflection $\Delta'_L$ | 0 | −42.90 | −67.26 | −75.88 |
| 10 | Resultant deflection $\Delta'_L$ | 0 | −0.032 | −0.050 | −0.057 |
| | | | | | Final |
| 11 | .... | 0 | −0.0326 | −0.0503 | −0.0570 |
| | | | | | Total Deflections |
| 12 | .... | 0 | 1.21 | 2.21 | 2.92 |

## Axial Dead Load ($\delta_a$) and Total Dead Load Deflections ($\Delta_D$)

| Tenth Points: 4 | 5 | 6 | 7 | 8 | 9 | 10 | Common factor |
|---|---|---|---|---|---|---|---|
| 11,834 | 11,834 | 11,834 | 11,834 | 7,098 | 3,387 | 5,071 | .... |
| $10\lambda = 125$ Ft | | | | | | | |
| −0.17 | −0.18 | −0.17 | −0.15 | −0.11 | −0.06 | 0 | .... |
| −4.36 | −4.16 | −3.95 | −3.72 | −2.79 | −2.03 | 0 | |
| −223.6 | −213.5 | −202.5 | −190.8 | −143.0 | −104.0 | 0 | $10^3$ |
| +18.87 | +17.99 | +17.08 | +16.09 | +20.15 | +30.72 | 0 | $1/E$ |
| | +16.75 | +34.74 | +51.82 | +67.91 | +88.06 | +118.78 | $\lambda/E$ |
| −378.06 | −361.31 | −326.31 | −274.75 | −206.84 | −118.78 | | $\lambda^2/E$ |
| +1.57 | +1.50 | +1.42 | +1.34 | +1.68 | +2.56 | | $\lambda^2/E$ |
| +134.35 | +111.96 | +89.56 | +67.17 | +44.78 | +22.39 | | $\lambda^2/E$ |
| 242.14 | 247.85 | 235.33 | 206.24 | 160.38 | 93.83 | 0 | $\lambda^2/E$ |
| 0.182 | 0.186 | 0.177 | 0.155 | 0.120 | 0.071 | 0 | |
| $\delta_a$ Leading to the Same Deflections $\delta'_a$: | | | | | | | |
| −0.182 | −0.186 | −0.177 | −0.155 | −0.121 | −0.070 | 0 | .... |
| Deflections $\Delta_D = \delta_w + \delta_a$ (Line 8, Table 3 + Line 11, Table 4): | | | | | | | |
| 3.42 | 3.58 | 3.44 | 3.02 | 2.32 | 1.31 | 0 | .... |

## Due to Live Load ($\Delta_L$) and Total Deflections ($\Delta$)

| Tenth Points: 4 | 5 | 6 | 7 | 8 | 9 | 10 | Common factor |
|---|---|---|---|---|---|---|---|
| 11,834 | 11,834 | 11,834 | 11,834 | 7,098 | 3,387 | 5,071 | |
| $10\lambda = 125$ Ft | | | | | | | .... .... |
| −0.036 | −0.019 | −0.016 | −0.026 | −0.017 | −0.031 | 0 | |
| −0.806 | −0.419 | −0.356 | −0.576 | −0.377 | −0.691 | 0 | |
| −52.4 | −27.2 | −23.1 | −37.5 | −24.5 | −44.9 | 0 | $10^3$ |
| +4.43 | +2.30 | +1.95 | +3.17 | +3.45 | +13.25 | 0 | $1/E$ |
| | +0.13 | +2.43 | +4.38 | +7.55 | +11.00 | +24.25 | $\lambda/E$ |
| −49.54 | −49.41 | −47.18 | −42.80 | −35.25 | −24.25 | 0 | $\lambda^2/E$ |
| +0.37 | +0.19 | +0.16 | +0.26 | +0.29 | +1.11 | 0 | $\lambda^2/E$ |
| −26.85 | −22.37 | −17.90 | −13.42 | −8.95 | −4.47 | 0 | $\lambda^2/E$ |
| −76.02 | −71.59 | −64.92 | −55.96 | −43.91 | −27.61 | 0 | $\lambda^2/E$ |
| −0.057 | −0.054 | −0.049 | −0.042 | −0.033 | −0.021 | 0 | |
| Deflections $\Delta_L$ Leading to the Same Deflections $\Delta'_L$ | | | | | | | |
| −0.0574 | −0.0534 | −0.0494 | −0.0425 | −0.0335 | −0.0211 | 0 | .... |
| $\Delta = \Delta_D + \Delta_L$ (Line 12, Table 4 + Line 11, Table 5) | | | | | | | |
| 3.36 | 3.53 | 3.39 | 2.98 | 2.29 | 1.29 | 0 | .... |

upward deflection, as the moment arm is not proportional to the deflection caused by the lateral loads. The writer believes that a fair approximation can be made in such cases by mentally comparing the ratio of the moments due to axial load and to lateral load and applying this ratio to the deflection caused by the lateral load. It is believed that even a quick comparison will yield an estimate which will give the correct answer in no more than two sets of computations. Table 5 shows the computations for $\Delta_L$ and the values of the total deflection $\Delta$.

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DISCUSSIONS

# ANALYTICAL AND EXPERIMENTAL METHODS IN ENGINEERING SEISMOLOGY

## Discussion

BY ROBERT E. GLOVER, ESQ.

ROBERT E. GLOVER,[52] ESQ.[52a]—The author develops a method of using a torsion pendulum to determine the amount of resonance to be expected when a structure of known frequency is subjected to an earthquake. This ingenious procedure makes use of an accelerometer record obtained during the quake. A torsion pendulum operation based on the use of a displacement curve, derived by double integration of an accelerometer record, has been used by the writer for a similar purpose. A comparison of results obtained by the two methods is shown in Table 5, which is based on the East-West component of the Helena quake of October 31, 1935.

TABLE 5.—COMPARISON OF ACCELERATION RESULTS

| Structure period (sec) | MAXIMUM ACCELERATION | |
|---|---|---|
| | Author's data[a] | Writer's data[b] |
| 0.20 | 0.73 *g* | 0.79 *g* |
| 0.25 | 0.67 *g* | 0.69 *g* |
| 0.40 | 0.50 *g* | 0.52 *g* |
| 0.50 | 0.32 *g* | 0.27 *g* |
| 0.80 | 0.19 *g* | 0.20 *g* |
| 1.00 | 0.24 *g* | 0.26 *g* |
| 2.00 | 0.10 *g* | 0.06 *g* |

[a] Scaled from author's spectrum curve. [b] Based on an integration made by the U. S. Coast and Geodetic Survey.

The slight differences are probably due to the choice of values for the constants of integration used in computing the displacement curve. Since the accelerometer instruments are generally arranged to be put in motion by a starter, which responds to ground movement, a small part of the record near the beginning is lost, and there is, therefore, some room for exercise of judgment when the values of these constants must be selected. The results obtained by the two methods appear, however, to be in essential agreement.

NOTE.—This paper by M. A. Biot, Esq., was published in January, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1942, by Messrs. George R. Rich, N. J. Hoff, and Merit P. White; May, 1942, by Messrs. N. H. Heck, Frank Neumann, and Jacob Feld; and June, 1942, by Messrs. George W. Housner, and Homer M. Hadley.

[52] Senior Engr., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

[52a] Received by the Secretary June 9, 1942.

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DISCUSSIONS

# HYDRODYNAMICS OF MODEL STORM SEWER INLETS APPLIED TO DESIGN

## Discussion

BY RALPH W. POWELL, M. AM. SOC. C. E.

RALPH W. POWELL,[7] M. AM. SOC. C. E.[7a]—A first reading of this paper gives one the impression that large caliber mathematical artillery has been brought to bear on a problem that should have succumbed to simpler methods. In default of a simpler method to suggest, perhaps such a criticism is ungracious; and this method of "distributional" analysis seems quite worth studying on its own account, entirely apart from the problem to which it is here applied. As the writer understands it, Mr. Tapley's method (omitting some of the details) is equivalent to the following. He has computed the kinematic viscosity of an imaginary, less viscous liquid (which would have produced Froudian similarity in his model) and the Reynolds number resulting from using the actual size and velocity of his model, but this imaginary viscosity instead of the actual viscosity. By running tests at several different slopes, he obtained enough variation in his mean velocities to give an appreciable variation in the Reynolds and Froude numbers and to derive a formula for the coefficient which he was studying in terms of **F** and **R**. His range of observed Reynolds numbers was only from 11,810 to 29,970, but by a very clever mathematical device (Eqs. 26 to 29), the validity of which the writer is not in a position to discuss at present, the range is extended to the entire field from zero to infinity. Therefore, the value of the coefficient for the aforementioned fictitious Reynolds number is obtained and the problem is solved. The writer wishes to make just a few observations.

It would seem that Eq. 6*b* is only one of the forms that Eq. 6*a* might take. For example, might not the equation be expressed as

$$K = C_{AD} [\alpha (\mathbf{F}) + \beta (\mathbf{R})] \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

Of course, the experimental results seem to answer this particular question in the negative, but in general it must not be assumed that $\theta$ (**F**, **R**) always can be put in the form $\alpha$ (**F**) $\beta$ (**R**).

NOTE.—This paper by G. S. Tapley, Assoc. M. Am. Soc. C. E., was published in March, 1942, *Proceedings*.

[7] Associate Prof. of Mechanics, Ohio State Univ., Columbus, Ohio.

[7a] Received by the Secretary July 20, 1942.

To the writer, the most interesting point of the paper is the use of a "momentum average" velocity instead of the usual "discharge average" in computing **F** and **R**—that is, instead of defining average velocity as discharge divided by area, the author has defined it as momentum divided by mass. This gives about 20% higher velocities in the examples of the paper. Since the ratio of $\bar{v}$ to $v'$ for the different columns of Table 1 is quite constant, it would seem that this innovation would hardly affect the final result in this particular problem. However, it raises the very interesting question as to whether, in general, it would be an improvement over the ordinary usage. It would seem that it could hardly be expected ever to replace, entirely, the ordinary definitions of **F** and **R** because a knowledge of the velocity distribution would be required before either could be computed. It also raises the question as to whether one should ever use an "energy average"—the square root of twice the kinetic energy divided by the mass.

The ratio $\frac{\bar{v}}{v'}$ has been variously denoted by $\beta$, $\alpha$, and $K_m$ by various authors. J. W. Johnson,[8] Assoc. M. Am. Soc. C. E., and M. P. O'Brien,[8,9] G. H. Hickox,[9] and R. W. Powell,[10] Members, Am. Soc. C. E., used $\beta$; S. M. Woodward, M. Am. Soc. C. E., and C. J. Posey,[11] Assoc. M. Am. Soc. C. E., used $\alpha$; and Hunter Rouse[12] and Victor L. Streeter,[13] Assoc. Members, Am. Soc. C. E., used $K_m$. The value $\beta = 1.20$, which is indicated in Table 1, is quite high, but the true value is still higher because, as the author admits, he has neglected the variation of velocity in the vertical direction. Since this would be considerable, the true value of $\beta$ was probably between 1.25 and 1.30—almost as much as for laminar flow. This must be due to the unsymmetrical cross section for, as the work of Professor Streeter,[13] shows, any reasonable assumption as to the frictional resistance would not account for such a high value if the section were symmetrical.

It should also be noted that in Eq. 8, and the other equations derived from it, special attention is required if any of the velocities are negative (that is, in the upstream direction). This has been pointed out by Professor Posey.[14]

A minor criticism of the paper is that the meaning of $h$ in the definitions of **F** and **R** is not clear to the casual reader. Another is that, in a model this small, it would seem that, among the factors listed in Eq. 3*a*, surface tension would have to be included. In fact, it is the writer's opinion (without any rational reason to back it up) that the model should have been at least twice as large—that is, $\frac{3}{16}$ scale instead of $\frac{3}{32}$ scale.

---

[8] "Velocity-Head Correction for Hydraulic Flow," by M. P. O'Brien and J. W. Johnson, *Engineering News-Record*, August 16, 1934, p. 215.

[9] "Applied Fluid Mechanics," by M. P. O'Brien and G. H. Hickox, McGraw-Hill Book Co., Inc., New York, N. Y., 1937, p. 271.

[10] "Mechanics of Liquids," by R. W. Powell, Macmillan Co., 1940, p. 125.

[11] "The Hydraulics of Steady Flow in Open Channels," by S. M. Woodward and C. J. Posey, John Wiley and Sons, Inc., New York, N. Y., 1941, p. 47.

[12] "Fluid Mechanics for Hydraulic Engineers," by Hunter Rouse, McGraw-Hill Book Co., Inc., New York, N. Y., 1938, p. 54.

[13] "The Kinetic Energy and Momentum Correction Factors for Pipes and for Open Channels of Great Width," by Victor L. Streeter, *Civil Engineering*, April, 1942, pp. 212–213.

[14] "Variations in Correction Factors for Pipes and Open Channels," by C. J. Posey, *ibid.*, July, 1942, p. 398.

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DISCUSSIONS

# BENDING MOMENTS IN THE WALLS OF RECTANGULAR TANKS

## Discussion

BY DANA YOUNG, M. AM. SOC. C. E.

DANA YOUNG,[15] M. AM. SOC. C. E.[15a]—The comparative data presented by Mr. Silverman in Table 2 are very interesting, and it is satisfying to see that the results are in reasonably close agreement. In drawing the curves 5(*a*) of Figs. 6 and 7, an attempt was made to provide design data that would be on the safe side. The significant fact is that the actual moments are considerably less than for a vertical or horizontal beam strip.

The calculation of plates under any except the simpler boundary conditions is at best a tedious task, and it is not expected that the detailed calculations should be made by every designer. Fortunately, it is not necessary to do so, any more than it is necessary for a designer to integrate the equation of the elastic curve every time he wishes to know the moments in a clamped beam. A table of moments for all useful cases is what is needed; the results given represent a start in this direction.

NOTE.—This paper by Dana Young, Assoc. M. Am. Soc. C. E., was published in November, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by I. K. Silverman, Assoc. M. Am. Soc. C. E.

[15] Prof. of Applied Mechanics, The Univ. of Texas, Austin, Tex.

[15a] Received by the Secretary August 17, 1942.